BWFD 2024 Day 1

Monday, June 3, 2024

TODAY:

✓ ① MODELS IN FORMAL DEMOGRAPHY

✓ ② THE MOST IMPORTANT EQN IN DEMOG
- ✓ → GROWTH MODELS
- ✓ → LEXIS DIAGRAMS / MODELS

✓ ③ GROWTH → NRR → DECOMPOSITION OF THE NRR

→ ④ PROJECTION ←

⑤ CONSEQUENCES OF PROJS W/ FIXED RATES

MODELS IN FORMAL DEMOGRAPHY

FORMAL DEMOGRAPHY WE LOOK RELATIONSHIPS BTW. CHARACTERISTICS

THE PURPOSE OF MODELING

GEORGE BOX: ~~"ALL MODELS ARE WRONG BUT SOME ARE USEFUL."~~

SAM KARLIN: "THE PURPOSE OF MODELS IS NOT TO FIT THE DATA BUT TO SHARPEN THE QUESTIONS."

ROBERT CHUNG: THE TEST OF MODELS IS HOW THEY FIT THE DATA.

BALANCING EQUATION

$$\text{POP}_{\text{TODAY}} = \text{POP}_{\text{YESTERDAY}} + \text{BIRTHS}_{\text{YESTERDAY}} - \text{DEATHS}_{\text{YESTERDAY}}$$

$$+ \cancel{\text{IN-MIGRANTS}_Y} - \cancel{\text{OUT MIGRANTS}_Y} + \cancel{\text{ADJ}_Y} + \cancel{\text{ERRORS}_Y}$$

"STOCK" CENSUS

"FLOW" VITAL REGISTRIES

$$\text{POP}_{\text{NOW}} = \text{POP}_{\text{BEFORE}} + \text{BIRTH}_{\text{INTERIM}} - \text{DEATHS}_{\text{INTERIM}}$$

$$K_{2024} = \text{POP IN 2024}$$

UPPERCASE NUMBERS

$$K_{2024} = K_{2023} + B_{2023} - D_{2023}$$

$$= K_{2023}\left(1 + \frac{B_{2023}}{K_{2023}} - \frac{D_{2023}}{K_{2023}}\right)$$

$$= K_{2023}\left(1 + b_{2023} - d_{2023}\right)$$

CRUDE BIRTH RATE 2023   CRUDE DEATH RATE 2023

$$b_{2023} - d_{2023} = \text{CRUDE RATE OF NATURAL INCREASE}_{2023}$$

$$= r_{2023}$$

$$b - d = r$$

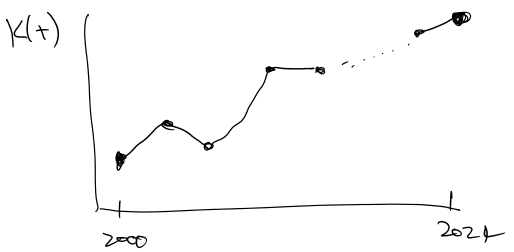
$$K_{2024} = K_{2023}(1 + r_{2023})$$

$$K_{2023} = K_{2022}(1 + r_{2022})$$

$$K_{2024} = K_{2022}(1 + r_{2022})(1 + r_{2023})$$

$$K_{2024} = K_{2000}\underbrace{(1 + r_{2000})(1 + r_{2001})(1 + r_{2002})\cdots(1 + r_{2023})}_{24 \text{ TERMS}}$$

MULTIPLICATIVE GEOMETRIC GROWTH



$$K_t = K_0(1 + r_0)(1 + r_1)(1 + r_2)\cdots(1 + r_{t-1})$$
$$= K_0 \prod_{i=0}^{t-1}(1 + r_i)$$

GEOMETRIC MULTIPLICATIVE GROWTH

WE HAVE r_i's.

WHAT IS THE AVERAGE r OVER THIS PERIOD?

$\bar{r}$ IS NOT TYPICALLY THE ARITHMETIC MEAN OF r_i's.

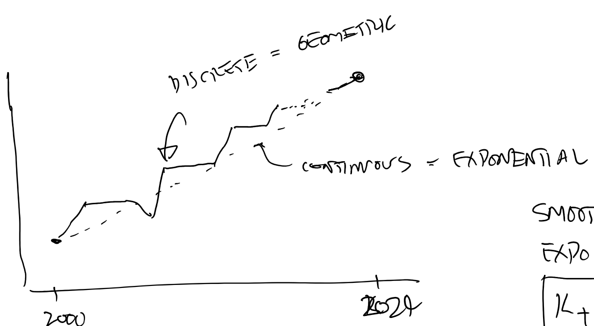
$$K_t = K_0 \prod_{i=0}^{t-1}(1 + r_i) = K_0 \prod_{i=0}^{t-1}(1 + \bar{r}) = K_0(1 + \bar{r})^t$$

$K_t = K_0(1 + \bar{r})^t$ THEN

$$\frac{K_t}{K_0} = (1 + r)^t \Rightarrow \left(\frac{K_t}{K_0}\right)^{\frac{1}{t}} = 1 + r$$

$$\Rightarrow \boxed{r = \left(\frac{K_t}{K_0}\right)^{\frac{1}{t}} - 1}$$

HOW WE FIND AVERAGE r GIVEN K_0, K_t FOR GEOMETRIC MODEL.



SMOOTH GROWTH MODEL

EXPONENTIAL GROWTH MODEL

$$\boxed{K_t = K_0 e^{rt}}$$

$$\frac{K_t}{K_0} = e^{rt}$$

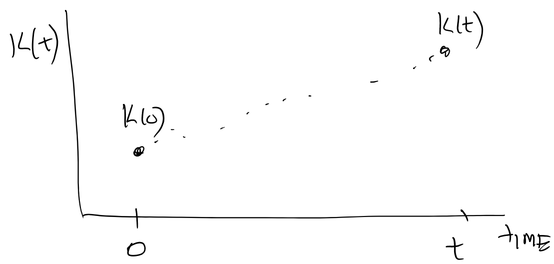
$$\log\left(\frac{K_t}{K_0}\right) = rt$$

$$\boxed{r = \frac{1}{t}\lg\left(\frac{K_t}{K_0}\right)}$$ EXPONENTIAL

AS LONG AS TIME INTERVAL t IS SHORT OR
r IS SMALL, GEOMETRIC AND EXPONENTIAL MODELS
ARE CLOSE.

FOR THIS WEEK, WE GET TO USE EITHER
GEOMETRIC OR EXPONENTIAL MODEL WHICHEVER IS EASIEST

DOUBLING TIMES



$$k(t) = 2\,k(0)$$
$$K(t) = K(0)\,e^{rt}$$

K_t, K_0 ← DISCRETE MULTIPLICATIVE GEOMETRIC

$K(t), K(0)$ ← CONTINUOUS EXPONENTIAL

$$\left.\begin{aligned} K(t) &= K(0)\,e^{rt} \\ K(t) &= 2\,K(0) \end{aligned}\right\} \quad 2 = e^{rt}$$

$$t_{DOUBLING} = \frac{\log 2}{r} = \frac{0.6931\ldots}{r}$$

SO IF r IS MEASURED IN % THEN $\frac{69.3}{r\%}$ = DOUBLING TIME.

RULE OF 72 FOR FINANCE PEOPLE IS
ACTUALLY THE RULE OF log 2

$$K_t = \tfrac{1}{2}K_0 \qquad \frac{K_t}{K_0} = e^{rt} = \frac{1}{2} \Rightarrow \log\left(\tfrac{1}{2}\right) = rt$$

Ddd

$$= \frac{-.6931}{r} = t_{HALF}$$

LONG-TERM GROWTH IN HUMAN POPULATIONS

LONG DOUBLING TIMES
r VERY CLOSE TO 0.
r HAS BEEN AT 0 FOR A REASONABLY LONG TIME
WE CALL THAT A STATIONARY POP

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$b = d$ $\quad d$ = DEATH RATE $= \frac{\text{\# DEATHS / YEAR}}{\text{POP}}$

$\frac{1}{d}$ = AVG TIME BETWEEN BIRTH AND DEATH

= EXPECTATION OF LENGTH OF LIFE

= LIFE EXPECTANCY $= e_0$ $\qquad e_x$

IN STATIONARITY $b = d$

$\frac{b}{d} = 1 \Rightarrow \boxed{b \cdot e_0 = 1}$ IN A STATIONARY POP.

STATIONARY POP IDENTITY

IN 2015 BOTH SEX $e_0 = 79.0 \approx 80$

$b = 12.5 / 1000 = 1.25\% = .0125$

$$\boxed{b_{US} \cdot e_{0\,US} \doteq 1}$$

(We broke for coffee here.)

GEOMETRIC MODEL : $K_t = K_0 (1+r)^t$

EXPONENTIAL MODEL : $K_t = K_0 e^{rt}$

$\Rightarrow K_t = K_0 A^t$

IF $A = 1+r \Rightarrow$ GEOMETRIC

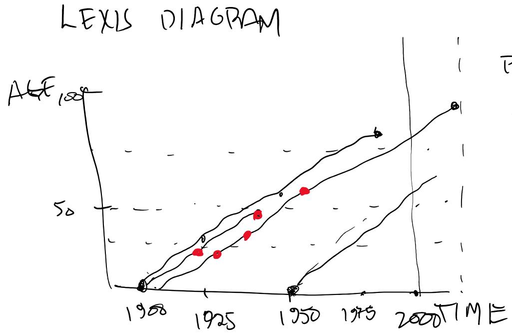
$A = e^r \Rightarrow$ EXPONENTIAL

UP TO NOW, WE WERE LOOKING AT TOTAL POP K_t

UNDIFFERENTIATED BY AGE, OR SEX, OR RACE, OR GEOGRAPHY.

NOW, I INTRODUCE AGE.

LEXIS DIAGRAM



EVERYONE GETS A LIFE LINE

SOME LINES ARE LONG

SOME UNFORTUNATELY ARE SHORT

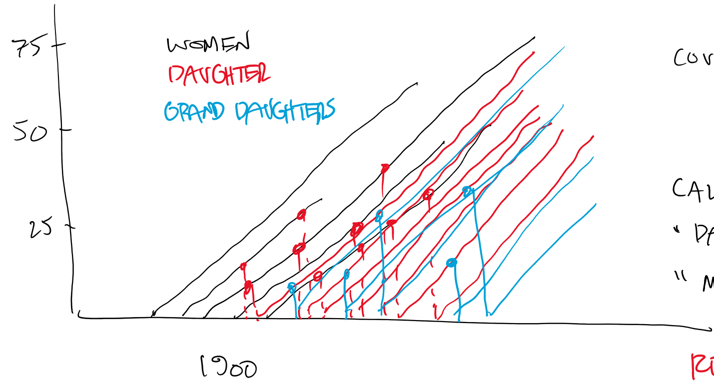
WOMEN

DIAGONAL LINES

GROUPS OF DIAGONAL LINES ARE COHORTS

HORIZONTAL LINES ARE AGES

VERTICAL LINES MARK PERIODS



COUNT UP THE DAUGHTERS
GRANDDAUGHTERS OF
THE ORIGINAL COHORT.

CALCULATE THE RATIO OF
"DAUGHTERS" IN ONE GENERATION TO
"MOTHERS" IN PREVIOUS GENERATION.

$\frac{\text{RED}}{\text{BLACK}} \quad \frac{\text{BLUE}}{\text{RED}} =$ RATIO OF FEMALES IN ONE GENERATION TO FEMALES IN PREVIOUS GEN.

NET REPRODUCTION RATIO = NRR

= R_0 IN IDE

R - NOUGHT

NAUGHT

$$\frac{K_t}{K_0} = e^{rt}$$

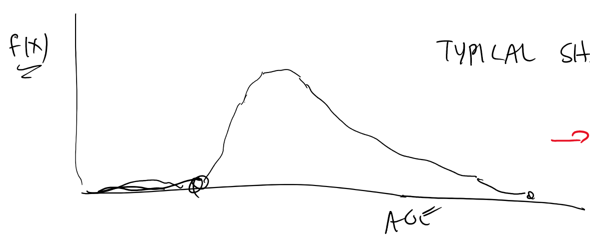
$$\frac{\text{FEMALES}_2}{\text{FEMALES}_1} = \boxed{NRR = e^{rG}}$$

WHERE G IS THE GENERATION LENGTH

$$\frac{\log NRR}{G} = r \qquad \frac{\log R_0}{G} = r$$

G TURNS OUT TO BE HARD TO CALCULATE

BUT CLAIM: G WILL BE CLOSE TO AVERAGE AGE AT WHICH WOMEN GIVE BIRTH (TO DAUGHTERS)



TYPICAL SHAPE OF CHILDBEARING BY AGE

$$\rightarrow \frac{\int x f(x)\,dx}{\int f(x)\,dx} = \text{MEAN AGE OF FERTILITY} = m$$

$$\rightarrow \frac{\sum (x+\frac{n}{2})\, {}_nf_x}{\sum {}_nf_x} \quad \text{FOR AGE GROUPS}$$

$$\frac{\int x f(x)\,dx}{\int f(x)\,dx} = m$$

$$\frac{\int x f(x)\, l(x)\,dx}{\int f(x)\, l(x)\,dx} = \mu$$

$$\frac{\int x\, f(x)\, l(x)\, dx}{\int f(x)\, l(x)\, dx} = \mu$$

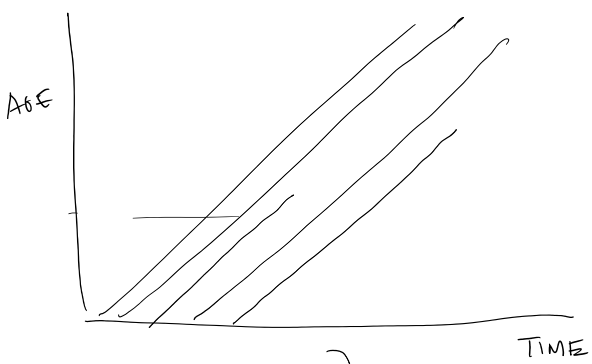
$$\frac{\int x\, f(x)\, l(x)\, e^{-rx}\, dx}{\int f(x)\, l(x)\, e^{-rx}\, dx} = \; ? \; = A_r$$

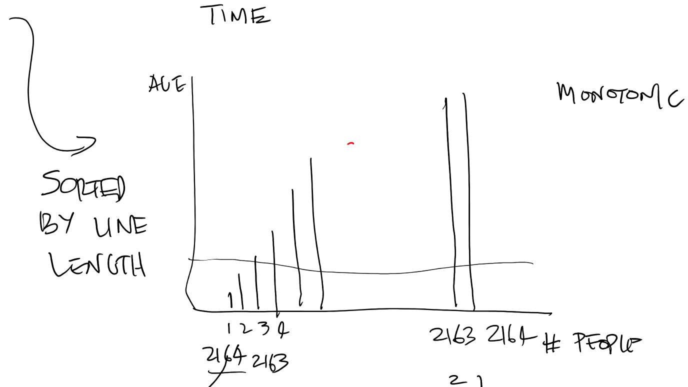
$\underline{f(x)}$ = AGE PATTERN FOR FERTILITY FOR CHILDREN OF BOTH SEXES.

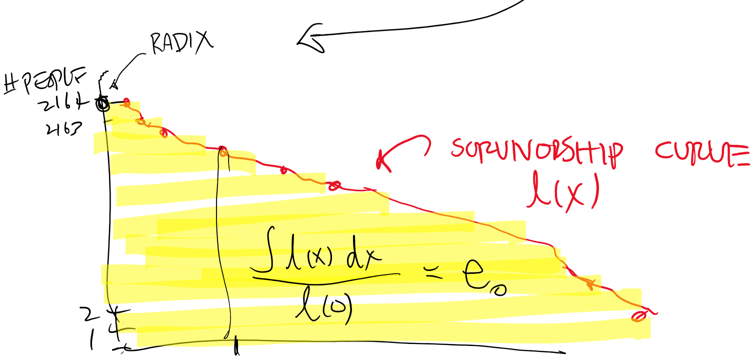
$f^{f}_{(x)}$

SEX RATIO AT BIRTH > 1 $\quad \frac{M}{F}$

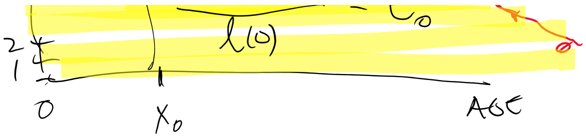
$\boxed{F_{FAB} = 0.4886}$ $\quad F_{MAB} = 0.5114 \quad \frac{.5114}{.4886} \doteq 1.046$







$$\frac{\int_0^{\omega} l(x)\, dx}{l(0)} = e_0$$



$$\frac{\circ}{l(0)} = e_0$$

$$\frac{\int_{x_0}^{\omega} l(x)\,dx}{l(x_0)} = e_x$$

Here is where we broke for lunch.

EXPONENTIAL GROWTH MODEL $K(t) = K(0)e^{rt}$

$$\frac{K(t)}{K(0)} = e^{rt} \Rightarrow \log\left(\frac{K(t)}{K(0)}\right) = rt \qquad r = \frac{1}{t}\log\left(\frac{K(t)}{K(0)}\right)$$

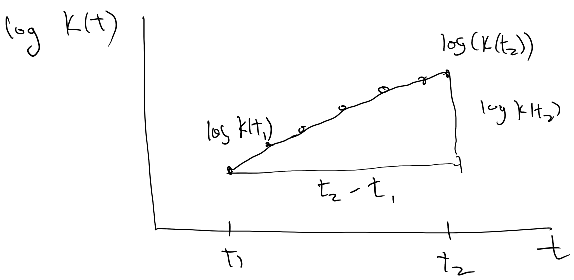
$$K(2044) = K(2020)\,e^{r \cdot 20}$$

PROJECTION WORKS LIKE WE'D EXPECT.

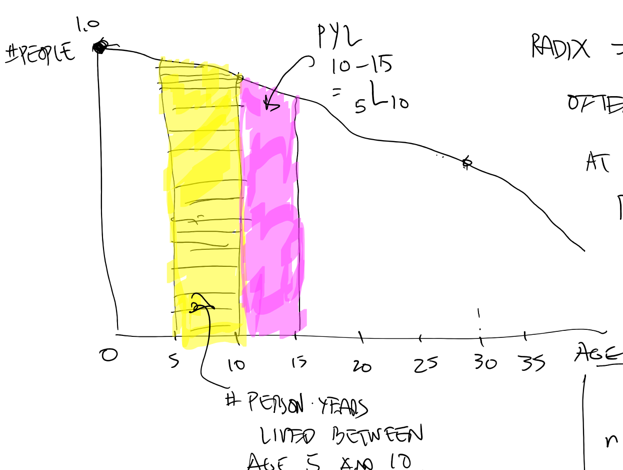
$$r = \frac{1}{t}\log\left(\frac{K(t)}{K(0)}\right) = \frac{1}{t}\left(\log K(t) - \log K(0)\right)$$

$$= \frac{1}{t_2 - t_1}\left(\log K(t_2) - \log K(t_1)\right)$$

$$= \frac{\log K(t_2) - \log K(t_1)}{t_2 - t_1}$$



DIAGNOSTIC AND ESTIMATION OF r FOR EXPONENTIAL GROWTH MODEL.
IF YOU PLOT $\log K(t)$ AGAINST t.
AND THE RELATIONSHIP IS LINEAR,
THEN THE SLOPE IS r
AND GROWTH IS EXPONENTIAL



RADIX $= l_0$ OR $l(0)$

OFTEN RADIX 1000, 100,000, 1,000,000

AT BERKELEY WE ALMOST ALWAYS SET RADIX AT 1.0.

$l(x)$ IS ALWAYS MONOTONIC

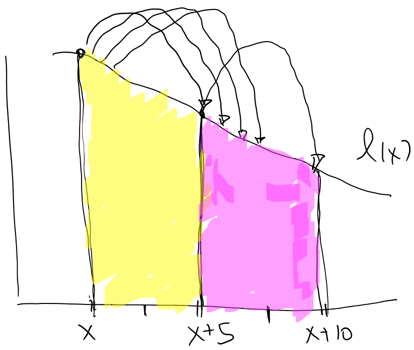
$$\boxed{{}_nL_x = \int_x^{x+n} l(y)\,dy}$$

$\int^{10}$

LIVED BETWEEN AGE 5 AND 10.

$$\int_5^{10} l(x)\,dx = {}_5L_5$$

$${}_nL_x = \int_x l(y)\,dy$$



$l(x)$ SURVIVORSHIP TO AN EXACT AGE X

$_nL_x$ AGE INTERVAL FROM X TO X+N

$\frac{l(x+5)}{l(x)}$ = RATIO OF SURVIVORS FROM AGE X TO X+5

CLAIM: IF WE ARE LOOKING AT THE SURVIVORSHIP PROBABILITY OF TO THAT WILL BE CLOSE

$$\boxed{\frac{{}_5L_{x+5}}{{}_5L_x} = }$$

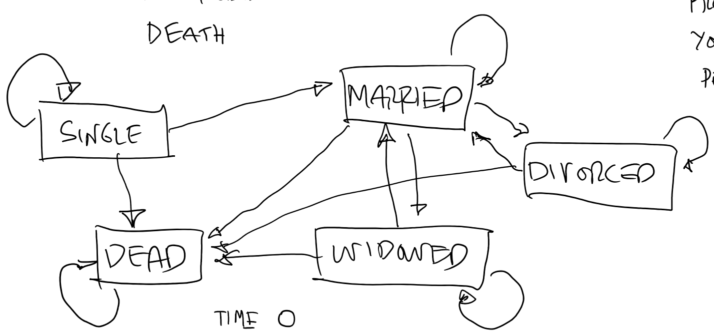
PROJECTIONS

MARITAL STATUS: 5 STATUSES

- SINGLE (NEVER MARRIED)
- MARRIED
- WIDOWED
- DIVORCED
- DEATH

FOR THIS MODEL:

- EXCLUSIVE
- COMPLETE
- PROJ PERIOD IS SHORT ENOUGH SO YOU CAN ONLY MAKE 1 TRANSITION PER PERIOD.



GRAPH THEORY

SOURCE

SINK OR ABSORBING STATE

TIME 0

TIME 1	S	M	W	DIV	DEAD
SINGLE	x	0	0	0	0
MARRIED	x	x	x	x	0
WIDOWED	0	x	x	0	0
DIVORCED	0	x	0	x	0
DEAD	x	x	x	x	1

$= \underline{A}$ 5x5

O's ARE STRUCTURAL ZEROES RATHER THAN OBSERVED ZEROES

K_t = COLUMN VECTOR OF COUNTS OF PEOPLE IN EACH STATUS AT TIME t.

K_0 IS OUR INITIAL POPULATION

$$K_1 = A_0 \cdot K_0$$
$$K_2 = A_1 \cdot K_1$$
$$K_3 = A_2 \cdot K_2$$

$$K_3 = A_2 \cdot A_1 \cdot A_0 \cdot K_0$$

IF A IS FIXED THEN $A_2 = A_1 = A_0$

$$K_3 = A^3 K_0$$

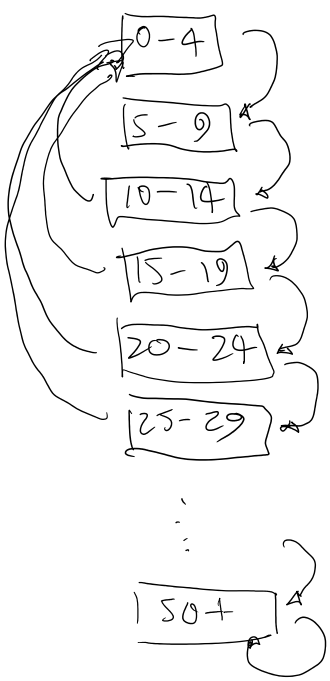
IN GENERAL $K_t = A^t K_0$

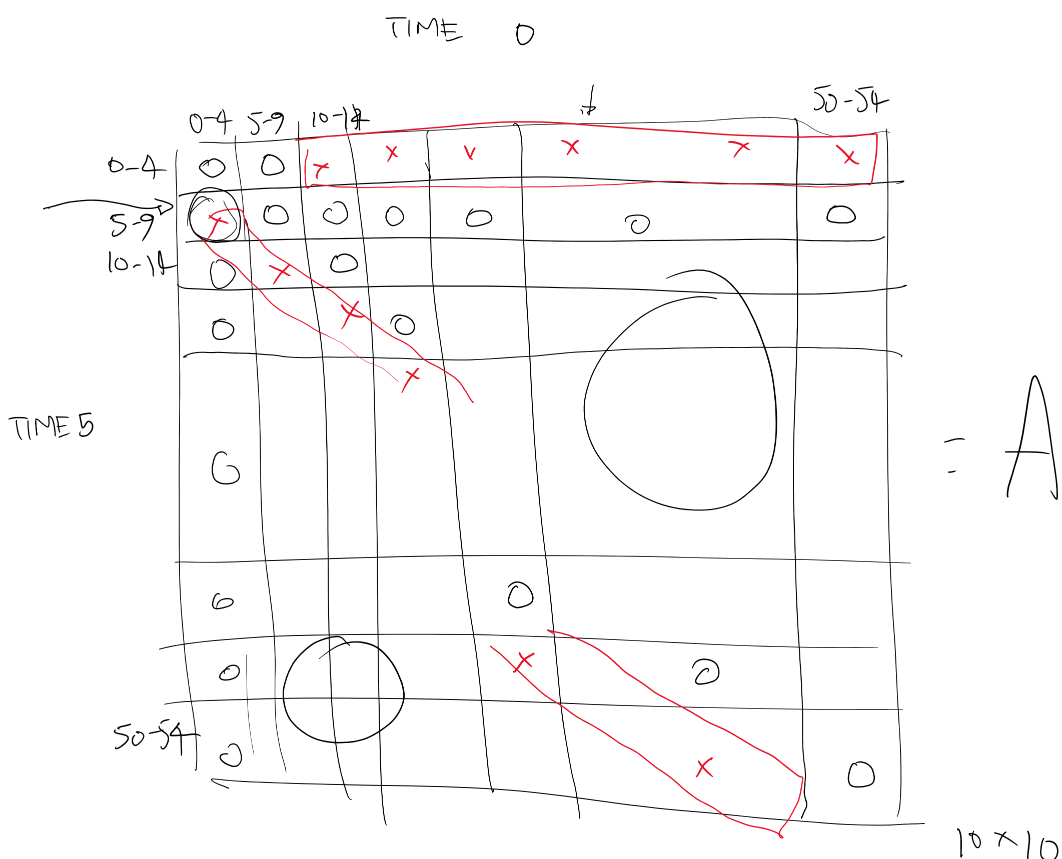
SPECIAL CASE OF TRANSITION MATRICES. (START WITH ARROW DIAGRAM)

5-YEAR AGE GROUPS

EACH STEP OF PROJECTION BE A 5-YEAR PROJECTION

FEMALE ONLY PROJECTION 10 AGE GROUPS





TWO PLACES W/O STRUCTURAL ZEROES:

1) TOP ROW (RELATED TO FERTILITY)

2) SUBDIAGONAL (RELATED TO SURVIVORSHIP FROM ONE AGE GROUP

1) TOP ROW (RELATED TO FERTILITY)

2) SUBDIAGONAL (RELATED TO SURVIVORSHIP FROM ONE AGE GROUP TO THE NEXT)

LESLIE MATRIX

SUBDIAGONALS : $\dfrac{{}_nL_{x+5}}{{}_nL_x}$

TOP ROW : BOOK

We did a little riff on the importance of publishing in well-known journals, noting that we ought To call these things Bernardelli matrices. Also a little riff on Galton-Watson stochastic branching processes and some things about Francis Galton; also how stochastic branching processes are an example of a statistical/mathematical model that went from demography to physics: the terms "critical mass," "subcritical process," and "supercritical process" come from Galton's model of the extinction of lineages and subsequently went on to be used to describe nuclear fission.

At this point, we did an extended example of projection with Leslie matrices to show that in the long run, the growth rate and age structure is independent of starting conditions and depends only on the Leslie matrix.