BWFD 2024 Day 2

Tuesday, June 4, 2024

YESTERDAY:

BALANCING EQUATION → GROWTH MODELS

MULTIPLICATIVE AND EXPONENTIAL

DISCRETE CONTINUOUS

AVERAGE GROWTH DEPENDS ON ENDPOINTS AND LENGTH OF INTERVAL

B, D, b, d, r

NRR IS R_0

$l(x)$ IS SURVIVORSHIP TO AGE x

${}_nL_x$ IS $\int_x^{x+n} l(y)\,dy$

TRANSITION MATRICES (IN GENERAL) AND ARROW DIAGRAMS

LESLIE MATRIX IS A SPECIAL TYPE OF TRANSITION MATRIX

A DEMONSTRATION OF PROJECTION WITH THE LESLIE MATRIX

CONSEQUENCES OF PROJECTION WITH FIXED LESLIE MATRIX

TODAY:

SOME CLEAN-UP / CLARIFICATIONS

REVISITING THE NRR AND r

STABILITY

OLD AGE DEPENDENCY

ADDING A PERSON TO THE PROJECTION

TOP-ROW ELEMENTS OF LESLIE MATRIX

SUB-DIAGONALS = SURVIVORSHIP RATIOS $\frac{{}_nL_{x+n}}{{}_nL_x}$

$$\boxed{\left(\frac{{}_nF_x + {}_nF_{x+n} \frac{{}_nL_{x+n}}{{}_nL_x}}{2} \right) \cdot \frac{{}_5L_0 \cdot F_{FAB}}{l_0}}$$

A

$K_1 = A K_0 \qquad K_2 = A K_1 \qquad K_3 = A K_2$

$K_3 = A \cdot A \cdot A \cdot K_0$ (MATRIX MULT NOT SCALAR)

$K_{50} = A^{50} K_0$ $\qquad A^{50}$ IS GOING TO BE COMPLETELY FILLED OUT

$K_{51} = A^{51} K_0$

THIS IS A DEMONSTRATION THAT THE CONVERGENCE OF THE AGE STRUCTURE IS ONLY ABOUT A AND NOT THE INITIAL CONDITIONS

ERGODICITY

Λ EIGENDECOMPOSITION OF A

ANY LESLIE MATRIX (AND MANY OTHER MATRICES TOO) CAN BE DECOMPOSED INTO A SPECIAL FORM.

$$A = U \Lambda U^{-1}$$

Λ IS DIAGONAL

U IS ORTHOGONAL

$$A \cdot A = U \Lambda \underbrace{U^{-1} U}_{} \Lambda U^{-1}$$

$$= U \Lambda \Lambda U^{-1} = U \Lambda^2 U^{-1}$$

$$K_t = A^t K_0 = U \underbrace{\Lambda^t}_{\text{DIAGONAL}} U^{-1} K_0$$

Λ IS COMPOSED OF DIAGONAL ELEMENTS EACH OF WHICH WE CALL EIGENVALUES λ_i

U IS COMPOSED OF EIGENVECTORS

FOR THE LESLIE MATRIX A WE USE FOR HUMAN POPULATIONS THERE WILL BE AN EIGENVALUE OF LARGEST MAGNITUDE AND IT WILL BE REAL AND TYPICALLY THERE WILL BE A BUNCH OF COMPLEX EIGENVALUES.

$x + iy$ $\qquad$ COMPLEX CONJUGATE PAIRS

OF COMPLEX EIGENVALUES.

$x + iy$
$x - iy$
COMPLEX CONJUGATE PAIRS

λ, u

$$\boxed{Au = \lambda u}$$

← THIS BEHAVIOR WE SAW YESTERDAY

MATRIX SCALAR

IT TURNS OUT THAT λ_1 IS RELATED TO THE GROWTH RATE

u_1 IS RELATED TO THE ULTIMATE LONG-RUN AGE DISTRIBUTION

eigen(A) ← SHORTCUT TO BRUTE-FORCING THE PROJECTION

K_t $K_0, K_1, K_2, \ldots K_{50}$ 10×1 COLUMN VECTORS
10 AGE GROUPS

${}_nK_x(t)$ = AGE GROUP x TO $x+n$ IN YEAR t.

${}_5K_0(2024)$ = AGE GROUP 0 TO 5 IN YEAR 2024

$K(x,t)$ CONTINUOUS VERSION EXACT AGE x IN YEAR t

$K(25, 2024\text{-}6\text{-}4)$

$K(25, 2024)$ = SURVIVORS OF BIRTHS FROM 1999.
= SURVIVORS OF $K(0, 1999)$

$$= \underbrace{K(0,1999)}_{\text{BIRTHS IN 1999}} \; \underbrace{l(25, 1999)}_{\text{PROPORTION SURVIVING TO AGE 25 FROM BIRTH COHORT 1999}}$$

BUT IF WE ARE IN A WORLD WITH CONSTANT A
THEN $l(25, 1999) = l(25)$ SAME FOR ALL YEARS

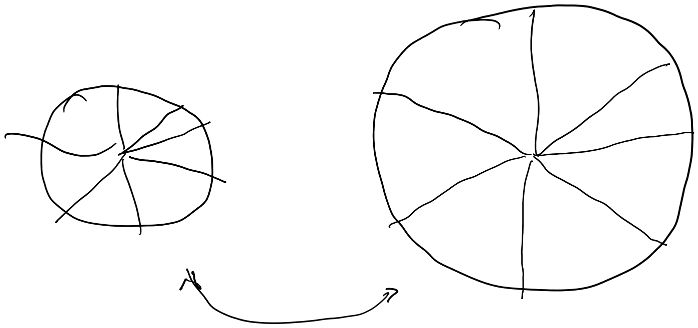
IN GENERAL

$$K(x,t) = K(0, t-x)\, l(x)$$
$$= B(t-x)\, l(x)$$

IN OUR WORLD WITH FIXED A, $f(x)$, $l(x)$ ARE FIXED

① WE KNOW AGE STRUCTURE IS FIXED.

② WE KNOW TOT POP IS GROWING LONG TERM AT RATE r.



THE ENTIRE PIE GROWS AT r

THE SLICES ARE FIXED AS A PROPORTION OF THE PIE

$\Big\}$ EACH SLICE GROWS AT r.

\# 25-YEAR OLDS GROW AT r

\# 70-YEAR OLDS GROW AT r

\# BABIES GROW AT r

$$B(2024) = B(2023)\, e^{r}$$
$$B(2024) = B(2022)\, e^{2r}$$
$$B(2024) = B(1999)\, e^{25r}$$

$$\underline{B(t) = B(t-x)\, e^{rx}} \Rightarrow B(t-x) = B(t)\, e^{-rx}$$

SO
$$K(x,t) = B(t-x)\, l(x)$$
$$= B(t)\, e^{-rx}\, l(x)$$

LET'S SAY $K(t) = \sum_x K(x,t)$ = TOTAL POP AT TIME t.

$$\frac{K(x,t)}{K(t)} = \frac{B(t)\, e^{-rx}\, l(x)}{K(t)}$$

PROPORTION OF POP AGE x AT TIME t $= b\, e^{-rx}\, l(x)$

$$\boxed{c(x) = b e^{-rx}\, l(x)}$$

THIS TELLS US WHAT THE LONG TERM EQUILIBRIUM

$$\boxed{c(x) = b e^{-rx} l(x)}$$

THIS TELLS US WHAT THE LONG TERM EQUILIBRIUM AGE STRUCTURE IS.

EULER FIGURED THIS OUT IN 1759

A STATIONARY POP IS A STABLE POP WHERE $r = 0$

$$\int c(x)\,dx = 1 = \int b e^{-rx} l(x)\,dx = b \int e^{-rx} l(x)\,dx$$

$$1 = b \int e^{-rx} l(x)\,dx \quad \text{SO}$$

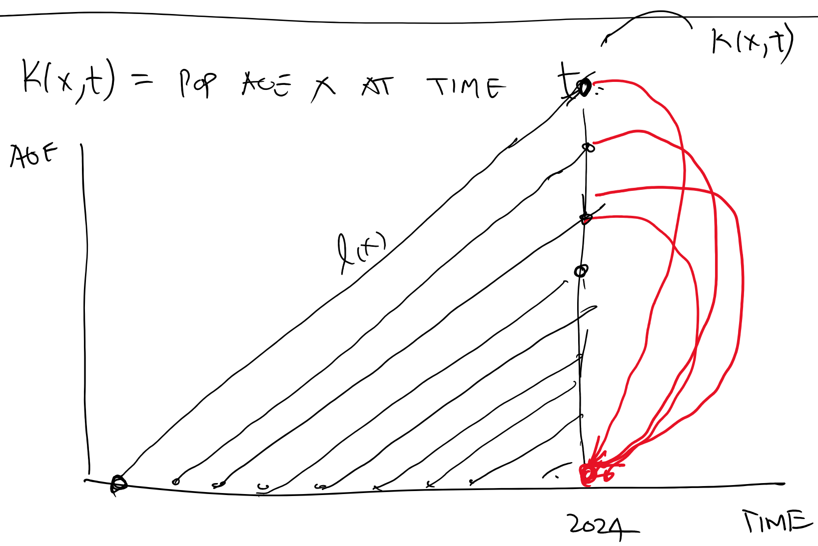
$$b = \frac{1}{\int e^{-rx} l(x)\,dx}$$

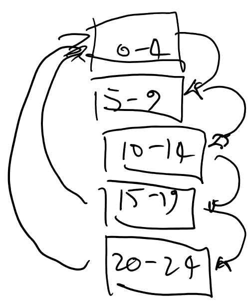
ALWAYS TRUE FOR STABLE POPULATIONS

IN A STATIONARY POP, $r = 0$ SO

$$b = \frac{1}{\int l(x)\,dx} = \frac{1}{e_0} \Rightarrow \boxed{b \cdot e_0 = 1}$$

$K(x,t)$ = POP AGE X AT TIME t





$$\underline{B(t)} = \int K(x,t) \cdot f(x)\,dx$$

$$= \int \underline{B(t-x)}\, l(x) f(x)\,dx$$

$$= \int B(t) e^{-rx} l(x) f(x)\,dx$$

$$B(t) = B(t) \boxed{\int e^{-rx}\, l(x)\, f(x)\, dx}$$

$$* \quad \boxed{\int e^{-rx}\, l(x)\, f(x)\, dx = 1} \quad *$$

EULER-LOTKA EQUATION

MEAN AGES IN GENERAL

MEAN AGE OF THE FERTILITY SCHEDULE

$$\bar{m} = A_m = \frac{\int x\, f(x)\, dx}{\int f(x)\, dx} = \frac{\sum (x+\frac{n}{2})\, {}_nF_x}{\sum {}_nF_x} \cdot \frac{\cancel{F_{FAB}}}{\cancel{F_{FAB}}}$$

WEIGHT $f(x)$

$$\boxed{\frac{\int x\, w(x)\, dx}{\int w(x)\, dx} = \text{FOR ANY WEIGHTED MEAN AGE}}$$

$$\mu = \frac{\int x\, f(x)\, l(x)\, dx}{\underbrace{\int f(x)\, l(x)\, dx}_{\text{NRR}}} = \frac{\sum (x+\frac{n}{2})\, {}_nF_x\, {}_nL_x}{\underbrace{\sum {}_nF_x\, {}_nL_x}_{\text{NRR}}}$$

$$A_r = \frac{\int x\, f(x)\, l(x)\, e^{-rx}\, dx}{\int f(x)\, l(x)\, e^{-rx}\, dx} = \text{MEAN AGE OF CHILDBEARING IN A STABLE POP GROWING AT } r$$

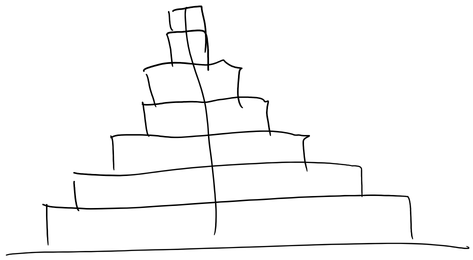
$$= \int x\, f(x)\, l(x)\, e^{-rx}\, dx$$

$$\bar{x}_r = \frac{\cancel{b}\int x\, l(x)\, e^{-rx}\, dx}{\ldots} = \text{MEAN AGE IN STABLE POP}$$

$$\bar{x}_r = \frac{\cancel{b}\int x\, l(x)\, e \quad dx}{\cancel{b}\int l(x) e^{-rx}\, dx} = \text{MEAN AGE IN STABLE POP}$$

SUPPOSE WE'RE IN A STABLE POP GROWING AT $r > 0$

WHAT DOES THE STABLE AGE DISTRIBUTION LOOK LIKE?

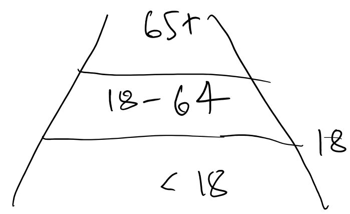


WHEN r IS HIGHER, WHAT DOES THIS LOOK LIKE?

$$c(x) = b\, l(x)\, e^{-rx}$$

BECAUSE OF $-rx$ IN EXPONENT, AS YOU GO UP IN AGE, A HIGHER r MAKES THE OLDEST AGES SMALLER.

WHAT HAPPENS TO OLD-AGE DEPENDENCY WHEN r CHANGES?



$$\frac{\text{POP } 65+}{\text{POP } 18\text{-}64} = \alpha = \text{OLD AGE DEPENDENCY RATIO}$$

$$c(x) = b \cdot l(x)\, e^{-rx} \qquad \text{RECALL THAT } b = \frac{1}{\int e^{-rx} l(x)\, dx}$$

$$c(x) = \frac{l(x)\, e^{-rx}}{\int e^{-rx} l(x)\, dx} \qquad \left[\, {}_nC_x = \frac{{}_nL_x\, e^{-rx}}{\sum {}_nL_x\, e^{-rx}} \,\right]$$

CONTINUOUS DISCRETE

WHAT IS α?

$$\alpha = \frac{\cancel{b}\int_{65}^{up} l(x)\, e^{-rx}\, dx}{\cancel{b}\int_{18}^{64} l(x)\, e^{-rx}\, dx}$$

WHAT HAPPENS TO α AS r CHANGES? $\frac{d\alpha}{dr}$

PRO TIP: IT IS OFTEN EASIER TO LOOK AT $\frac{d \log \alpha}{dr}$

$$\log \alpha = \log \left[\int_{65}^{\infty} l(x)\, e^{-rx}\, dx \right] - \log \int^{\infty} l \quad e^{-rx} \quad \log \int^{64} l(x)\, e^{-rx}\, dx$$

$$\log \alpha = \log\left[\frac{\int_{65}^{\infty} l(x)e^{-rx}\,dx}{\int_{18}^{64} l(x)e^{-rx}\,dx}\right] = \log \int_{65}^{\infty} l(x)e^{-rx}\,dx - \log \int_{18}^{64} l(x)e^{-rx}\,dx$$

$$\frac{d\log\alpha}{dr} = \frac{d}{dr}\log\int_{65}^{\infty} l(x)e^{-rx}\,dx - \frac{d}{dr}\log\int_{18}^{64} l(x)e^{-rx}\,dx$$

$$= \frac{\frac{d}{dr}\int_{65}^{\infty} l(x)e^{-rx}\,dx}{\int_{65}^{\infty} l(x)e^{-rx}\,dx} - \frac{\frac{d}{dr}\int_{18}^{64} l(x)e^{-rx}\,dx}{\int_{18}^{65} l(x)e^{-rx}\,dx}$$

$$= \frac{\int_{65}^{\infty} -x\, l(x)e^{-rx}\,dx}{\int_{65}^{\infty} l(x)e^{-rx}\,dx} - \frac{\int_{18}^{64} (-)x\, l(x)e^{-rx}\,dx}{\int_{18}^{64} l(x)e^{-rx}\,dx}$$

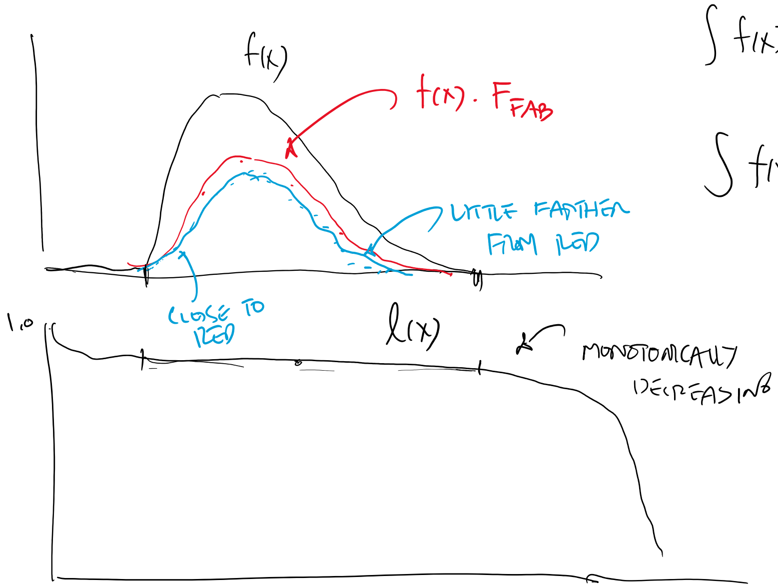
$$= \underbrace{\frac{\int_{18}^{64} x\, l(x)e^{-rx}\,dx}{\int_{18}^{64} l(x)e^{-rx}\,dx}}_{\text{AVERAGE AGE OF THE WORKING POPULATION}} - \underbrace{\frac{\int_{65}^{\infty} x\, l(x)e^{-rx}\,dx}{\int_{65}^{\infty} l(x)e^{-rx}\,dx}}_{\text{AVERAGE AGE OF RETIREES}}$$

$$\frac{d\log\alpha}{dr} = \bar{x}_{\text{workers}} - \bar{x}_{\text{RETIREES}}$$

~ 40 ISH $\qquad \sim 80$ ISH

-35 TO -40

NEGATIVE SIGN MEANS A 1% INCREASE IN GROWTH ⇒ 35 – 40%
DECREASE IN OLD-AGE DEPENDENCY RATIO
OR A 1% DECREASE IN GROWTH ⇒ 35-40% INCREASE IN α.



$\int f(x)\,dx = TFR$ TOTAL FERTILITY RATE

"TOTALLING UP"

$\int f(x)\,dx \cdot F_{FAB} = GRR$

GROSS REPRODUCTION RATE

$= TFR \cdot F_{FAB}$

$\int f(x)\,l(x)\,dx \cdot F_{FAB}$

$= NRR$

CLAIM: THERE IS SOME NUMBER THAT I CAN MULTIPLY THE AREA UNDER THE RED LINE BY TO GET THE AREA UNDER THE BLUE LINE.

MY CLAIM IS THAT IS GOING TO BE CLOSE TO $l(\mu)$

TFR

$TFR \cdot FFAB = GRR$

$TFR \cdot F_{FAB} \cdot l(\mu) = GRR \cdot l(\mu) = NRR$

$NRR = e^{rG}$

$\boxed{G = \frac{\log NRR}{r}}$

G TENDS TO BE PRETTY CLOSE TO μ.

$\frac{\log NRR}{G} = r$

HOLDING f(x) l(x) SHAPE CONSTANT WE COULD DELAY OR ADVANCE THE TIMING WHICH CHANGES G.