Lifetable Yearsaving

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This session is a walk-through of "Vaccinating the oldest..."
but more than a walk-through. It is also a work-out with
the technique behind the proof in the paper,
a technique called the "Big-H Assist"
with many applications to advanced lifetable analysis.

1. A Looming Ethical Dilemma
2. The Lifetable Yearsaving Function and COVID Vaccination
3. Monotone Increasing Yearsaving
4. The Big-H Assist
5. Prospects and Questions

References

Joshua R. Goldstein, Thomas Cassidy, and Kenneth W. Wachter. Vaccinating the oldest-old against COVID-19 saves both the most lives and the most years of life. *PNAS*, 118(11), 16 March 2021.

Kate Bubar, Kyle Reinholt, Stephen M. Kissler, Marc Lipsistch, Sarah Cobey, Yonatan H. Grad, and Daniel B. Larremore. Model-informed COVID-19 vaccine prioritization strategies by age and serostatus. *Science*, 371(6532):916–921, 26 February 2021.

Marcia C. Castro and Burton Singer. Prioritizing COVID-19 vaccination by age. *PNAS*, 118(15):1–2, 13 April 2021.

Milton Abramowitz and Irene Stegun. *Handbook of mathematical functions, with formulas, graphs, and mathematical tables*. National Bureau of Standards, U.S. Government Printing Offce, Washington, DC, 1964.

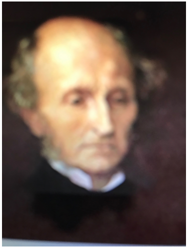
Lives Saved and Years of Life Saved

- You have a thousand doses of vaccine.
- You can vaccinate a thousand 90-year-olds or a thousand 50-year-olds.
- A thousand doses given to 90-year-olds will certainly save more lives than a thousand doses given to 50-year olds, since more 90-years olds will otherwise die.
- But, a 90-year-old, if saved from COVID, can look forward to about 4 years of remaining life, whereas a 50-year-old, if saved from COVID, can look forward to about 32 years of remaining life.
- Depending on how big the gradient in age-specific COVID mortality is, it could easily turn out that vaccinating the old could saves more lives but vaccinating the young could save more years of life.

Every life is equally precious.

Years of life have utility and value.

John Stuart Mill



Usually there are tradeoffs. Choices between the principles are fought out in courts of law in lawsuits about compensation for death and injury caused by negligence.
But we would be freed from struggling with ethical philosophy if it turned out that maximizing lives saved also maximizes years of life saved.

- The Lifetable Yearsaving Function is the Product of Hazard and Life Expectancy

$$h(x)\, e(x)$$

- We express person-years of life saved per vaccination of an x-year old, $v_e(x)$, by multiplying together
  - $\theta(x)$, Vaccine Efficacy, for COVID-19 fairly independent of age,
  - $\rho(x)$, the ratio of COVID mortality to all-other-cause mortality, initially close to constant because COVID mortality was observed to be to a proportional hazard,
  - $h(x)$ the age-specific hazard rate for all-other-cause mortality,
  - $e(x)$, future life expectancy at age x.
- In the paper, Equation 1 reads $v_e(x) = \theta \rho h(x) e(x)$, a quantity proportional to the lifetable yearsaving function.

Ask Later

- Direct effects: Save the lives of people you vaccinate.
- Indirect effects: Vaccinate younger super-spreaders and save other people's lives.
- Period $h(x)$ and Cohort $e(x)$.
- COVID mortality not quite proportional to All-Other-Cause Mortality
- Per person, not per age group measures.
- Higher-risk subgroups
- The changing picture as vaccination progresses.

3. Monotone Increasing Yearsaving

"Monotone" means always in the same direction, e.g. going up.

- Empirical curves for the U.S., Germany, and South Korea in the paper. The COVID death rate goes up faster than remaining life expectancy is going down, making yearsaving go up.
- Marcia Castro and Burt Singer (2021) show the same for provinces of Brazil. Arolas and colleagues at Rostock show the same for most of the countries in a sample of 40, but find a dozen exceptions, not all of which may be due to bad data.
- The empirical analysis suggests that increasing yearsaving is a fairly general property of realistic adult lifetables. The paper presents a mathematical condition on hazard rates which is sufficient to guarantee monotone increasing lifesaving.

A Mathematical Sufficient Condition

Proposition: Suppose the hazard $h(x)$ is twice continuously differentiable and for all ages x

$$\left(\frac{d}{dx}\log(h(x))\right)^2 > \frac{d^2}{dx^2}\log(h(x))$$

Then the yearsaving function $g(x) = h(x)e(x)$ is monotone increasing for all x.

Corollary: When the second derivative of the log hazard is zero or negative at ALL ages, yearsaving is monotone increasing.

Notation

- $H(x) = \int_0^x h(y)\,dy$ is the cumulative hazard;
- $\ell_x = \exp(-H(x)$ is lifetable survivorship;
- $e(x) = (1/\ell_x)\int_0^x \ell_y dy$ is remaining life expectancy;
- $g(x) = h(x)\,e(x)$ is lifetable yearsaving.

Examples of the Sufficient Condition in Action

Gompertz – For those who love calculus, differentiate.

$$h(x) = \alpha\, e^{\beta x}$$

$$\log h(x) = \log(\alpha) + \beta\, x$$

What is the second derivative (the slope of the slope) of the log hazard?

Logistic – For those who love computation, let's draw the curve.

$$h(x) = \frac{1}{1 + \exp(-x)}$$

Switch to sample R-code.

R-Code for Examining Logistic Log Hazard

```
# Section 1: Shape of the Logistic Log Hazard
#       Compute the Logistic Hazard and Cumulative Hazard
#       as functions of age, and observe the downward-bending
#       shape of the log hazard as a function of age.
#       This special case is stylized to make formulas simple.
#       For greater realism, picture one unit in x standing
#       for ten years of age, starting around age 60,
#       for a somewhat frail subset of the population.
  dx <- 0.01 # step size for age variable
  x <- seq(0,8, by = dx) # age variable
  h <- 1/(1 + exp(-x)) # hazard
  H <- log( (1 + exp(x))/2) # cumulative hazard
  ell <- exp(-H) # survivorship
# Now observe the downward bending of the log hazard
  plot( x, log(h), type = "l", xlim = c(0,4) )
  title( main = "Log Hazard as Function of Age" )
#       For the logistic, the slope of the log hazard is visibly decreasing over the
whole range, so the second derivative of the log hazard is always negative, and
the sufficient condition for monotone increasing yearsaving is satisfied.
```

4. The Big-H Assist

- The word "assist" is drawn from sports. When a player scores a goal with help from another player, the help is called an "assist".
- The goal here is proving that yearsaving increases with age. The Big-H Assist helps by expressing integrals in an easier form.
- The Big-H Assist is a Change of Variables from age x to cumulative hazard H.
- Simple but versatile, the Big-H Assist has many uses in advanced formal demography.

The Integral for Yearsaving

$$
\begin{aligned}
g(x) = h(x)e(x) &= h(x)\,\frac{1}{\ell_x}\int_x^\infty \ell_y\,dy \\
&= \frac{h(x)}{e^{-H(x)}}\int_x^\infty e^{-H(y)}\,dy
\end{aligned}
$$

$$
g(x) = \int_x^\infty h(x)e^{-(H(y)-H(x))}\left(\frac{1}{h(y)}\frac{dH(y)}{dy}\right)dy
$$

$$
\left(\frac{1}{h(y)}\frac{dH(y)}{dy}\right) = 1
$$

The Big-H Assist for Yearsaving

$$g(x) = \int_x^\infty h(x) e^{-(H(y)-H(x))} \left(\frac{1}{h(y)} \frac{dH(y)}{dy} \right) dy$$

Put

$$\begin{aligned} H(x) &= u \\ H(y) - H(x) &= t \\ h(x) &= \lambda(u) \\ h(y) &= \lambda(u+t) \\ X(H(x)) &= x \end{aligned}$$

Then

$$g(X(u)) = \int_0^\infty e^{-t} \left(\frac{\lambda(u)}{\lambda(u+t)} \right) dt$$

Section 2: The Hazard as a Function of the Cumulative Hazard
Observe the change in horizontal separations

Section 3: Introducing the "Big-H Assist"
We want values of h at a set of values of H that are
equally spaced, so that we can compute areas as sums.
Use the approx function in R to go from a chosen value
of H to the corresponding value of h. We call it lambda.

```
# Section 4. Evaluate lambda over a grid of values of t
#       For a trial value of u, plot the curve for the integrand.
#       The area under the curve is the value of yearsaving
#       for the age corresponding to this value of u.
```

```
# Section 5: Plot the ratio lambda(u)/lambda(u+t)
#       as a function of t for several values of u.
#       Observe that a higher value of u determines
#       a ratio cuver that is higher AT EVERY VALUE OF t.
```

Gompertz Hazards

Since $H(x) = (1/\beta)(\alpha e^{\beta x} - \alpha)$, we have $\lambda(u) = \alpha + \beta\, u$

$$g(X(u)) = \int_0^\infty e^{-t} \left(\frac{\alpha + \beta\, u}{\alpha + \beta\, u + \beta t} \right) dt$$

Now when u gets bigger, the fraction in brackets

$$(\ldots) = \left(\frac{1}{1 + \beta\, t/(\alpha + \beta\, u))} \right) dt$$

gets bigger *for every* t. For the Gompertz, that proves that $g(X(u)$ is a monotone increasing function of u and therefore $g(x)$ is a monotone increasing function of x.

Logistic Hazards

Since $H(x) = \log((1 + \exp(x))/2$, we have $\lambda(u) = 1 - (1/2)e^{-u}$.

$$g(X(u)) = \int_0^\infty e^{-t} \left(\frac{1 - (1/2)e^{-u}}{1 - (1/2)e^{-u}e^{-t}} \right) dt$$

Put $\gamma = (1/2)e^{-u}$. As u gets bigger, γ gets smaller, and the fraction in brackets

$$(\ldots) = \left(\frac{1}{1 - (\gamma/(1-\gamma))(1 - e^{-t})} \right) dt$$

gets bigger *for every* t. For these Logistic Hazards, that proves that $g(X(u))$ is a monotone increasing function of u and therefore $g(x)$ is a monotone increasing function of x.

Optional Calculus for the General Case

$$\frac{d}{du}\int_0^\infty e^{-t}\left(\frac{\lambda(u)}{\lambda(u+t)}\right)dt = \int_0^\infty e^{-t}\left(\frac{\lambda(u)}{\lambda(u+t)}\right)\left[\frac{d}{du}\log\left(\frac{\lambda(u)}{\lambda(u+t)}\right)\right]dt$$

We seek conditions that force the expression in square brackets to be positive for all t.

$$[\ldots] = \frac{d}{du}\log\lambda(u) - \frac{d}{du}\log\lambda(u+t)$$

We apply the Fundamental Theorem of Calculus by differentiating and then integrating back up.

$$[\ldots] = \int_0^t (-\frac{d}{d\tau}\frac{d}{du}\log(\lambda(u+\tau)))\,d\tau$$

Optional Completion of Proposition Proof via Calculus

Put $f(x) = \log h(x)$ so that $\log(\lambda(u)) = f(X(u))$. Then

$$-\frac{d}{d\tau}\frac{d}{du}f(X(u+\tau)) = -\frac{d^2}{dv^2}f(X(v))|_{u+\tau}$$

At y such that $H(y) = u + \tau$, this second derivative is equal to

$$\exp(-2f(y))\left\{\left(\frac{df(y)}{dy}\right)^2 - \frac{d^2f(y)}{dy^2}\right\}$$

So long as the quantity in curly brackets is always positive, the derivative of $g(X(u)$ with respect to u is always positive, so $g(x)$ is an increasing function of x, as the Proposition claims.

- The Big-H Assist comes into many other derivations in formal demography.
  - A formula for Gompertz life expectancy in terms of the Exponential Integral special function.
  - An easy derivation of the Lifetable Entropy for the Gompertz case.
  - Counterexamples to Monotone Increasing Yearsaving for unrealistic but mathematically interesting cases.
- Broader issues about COVID and Lifesaving

  e.g. Indirect effects, higher-risk subgroups, the changing picture as vaccination progresses. ,