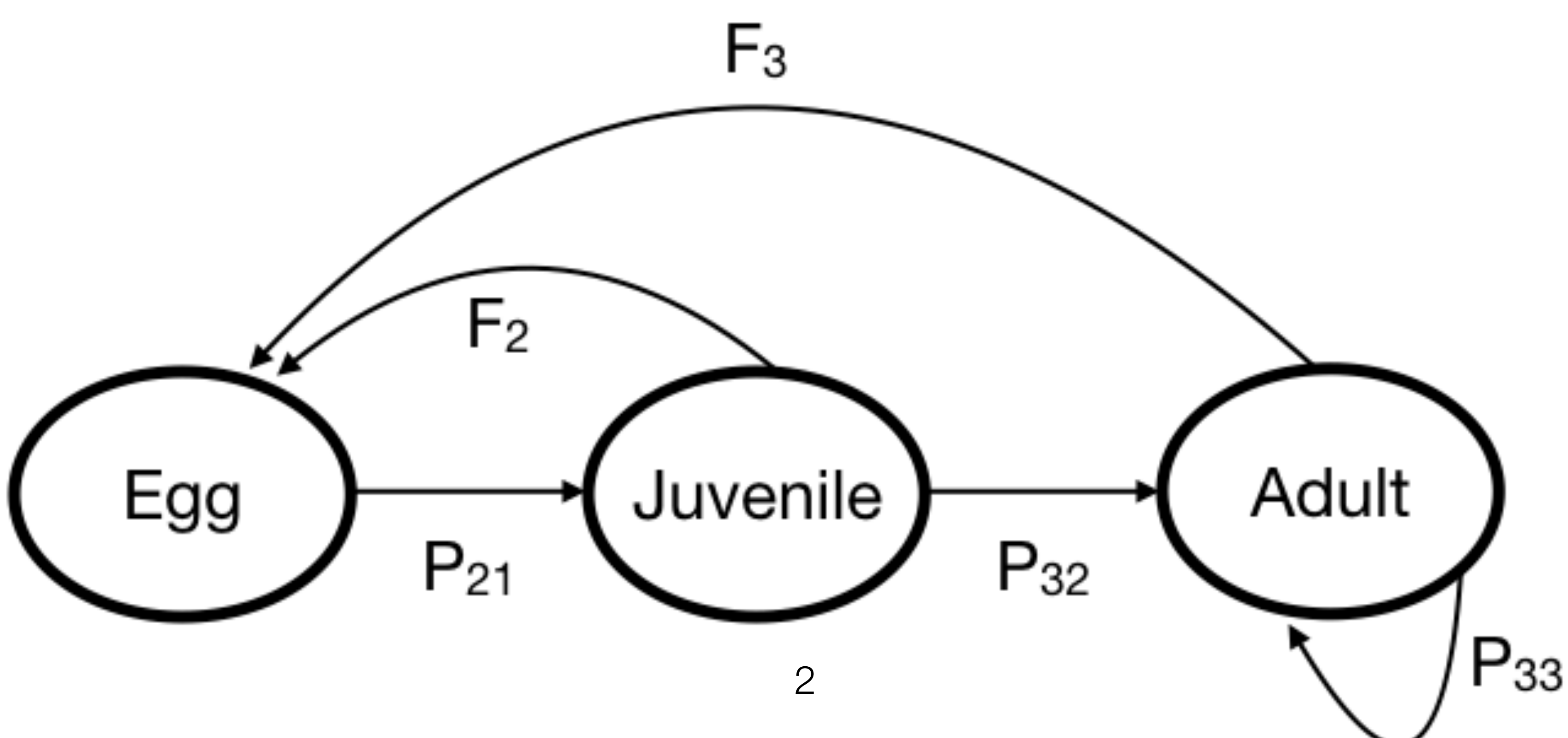
Berkeley Formal Demography Workshop 2021

Day 1: Population dynamics and stable population theory

Quick recap of pre-workshop exercises

Fish population with three life stages: Egg, Juvenile, Adult

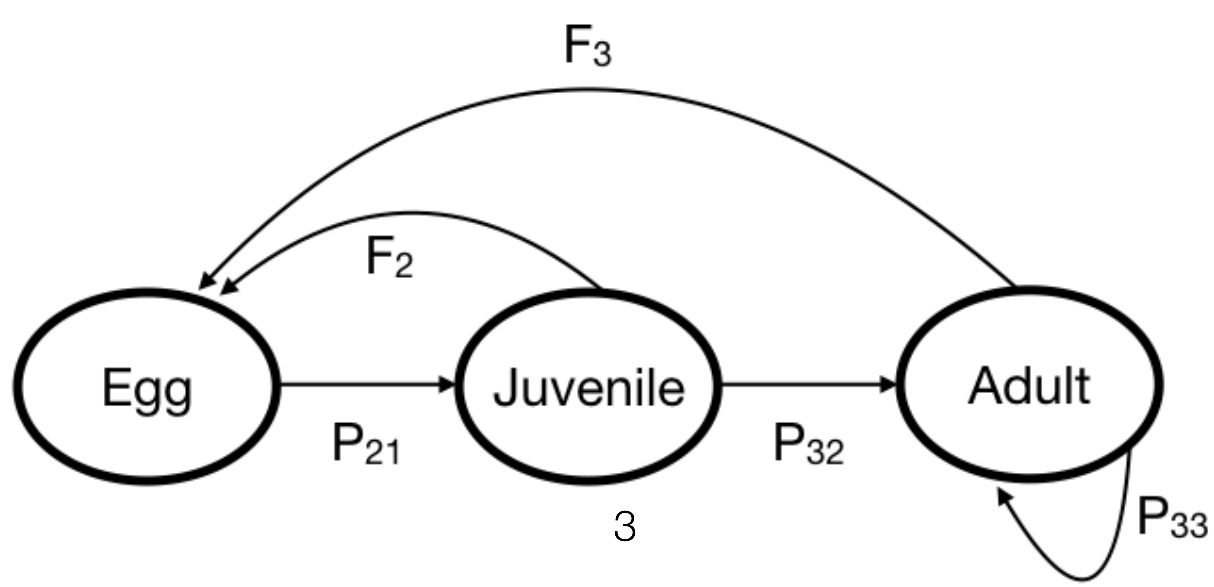
We assumed constant stage (or age) specific mortality and fertility rates and projected forward for 10 years.



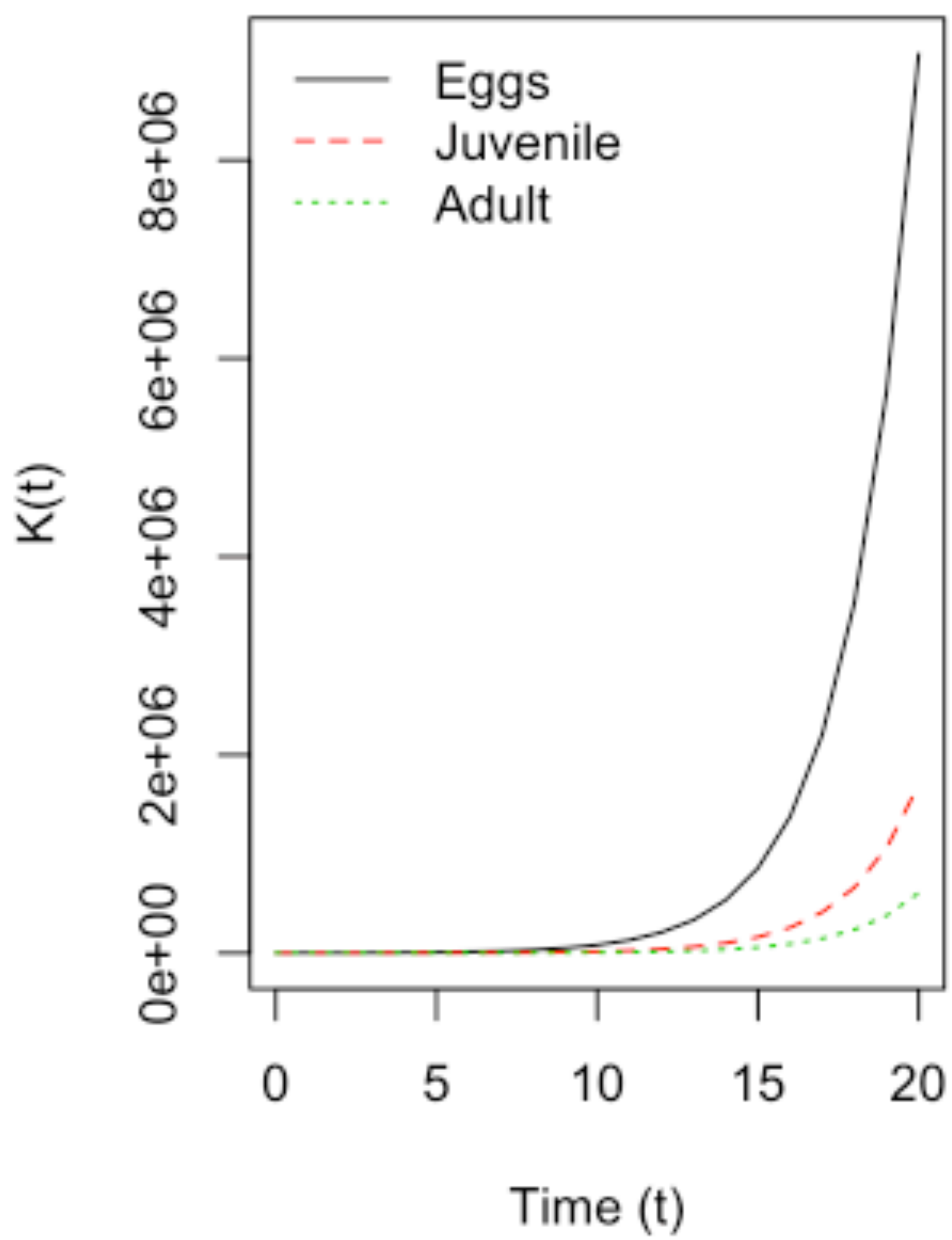
Quick recap of pre-workshop exercises

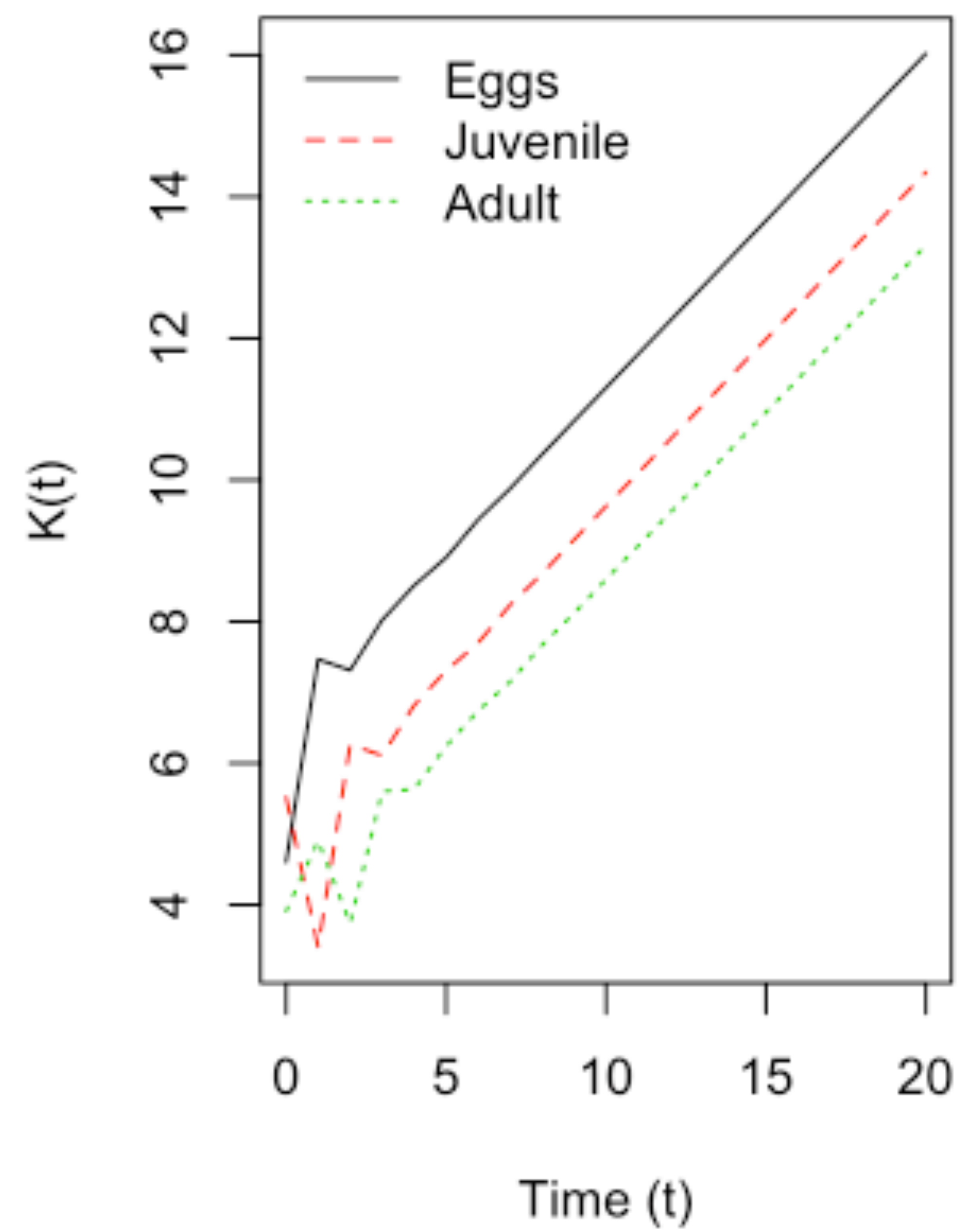
Define the **Leslie matrix** which describes the transition of the fish population from one stage to the next in a discrete (one year in our example) time step:

$$A = \begin{bmatrix} 0 & F_2 & F_3 \\ P_{21} & 0 & 0 \\ 0 & P_{32} & P_{33} \end{bmatrix} = \begin{bmatrix} 0 & 5 & 10 \\ 0.3 & 0 & 0 \\ 0 & 0.5 & 0.2 \end{bmatrix}$$

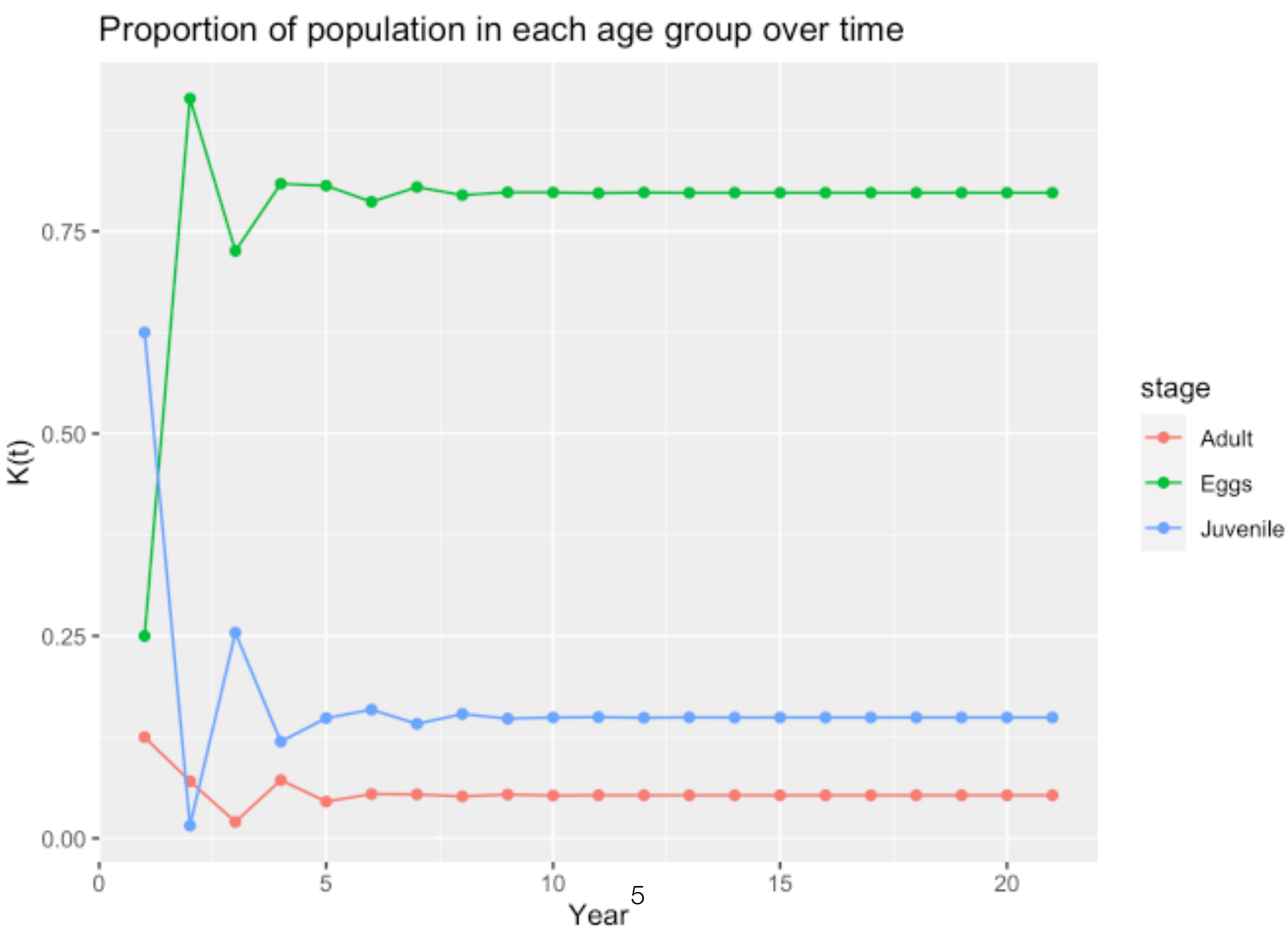


Quick recap of pre-workshop exercises

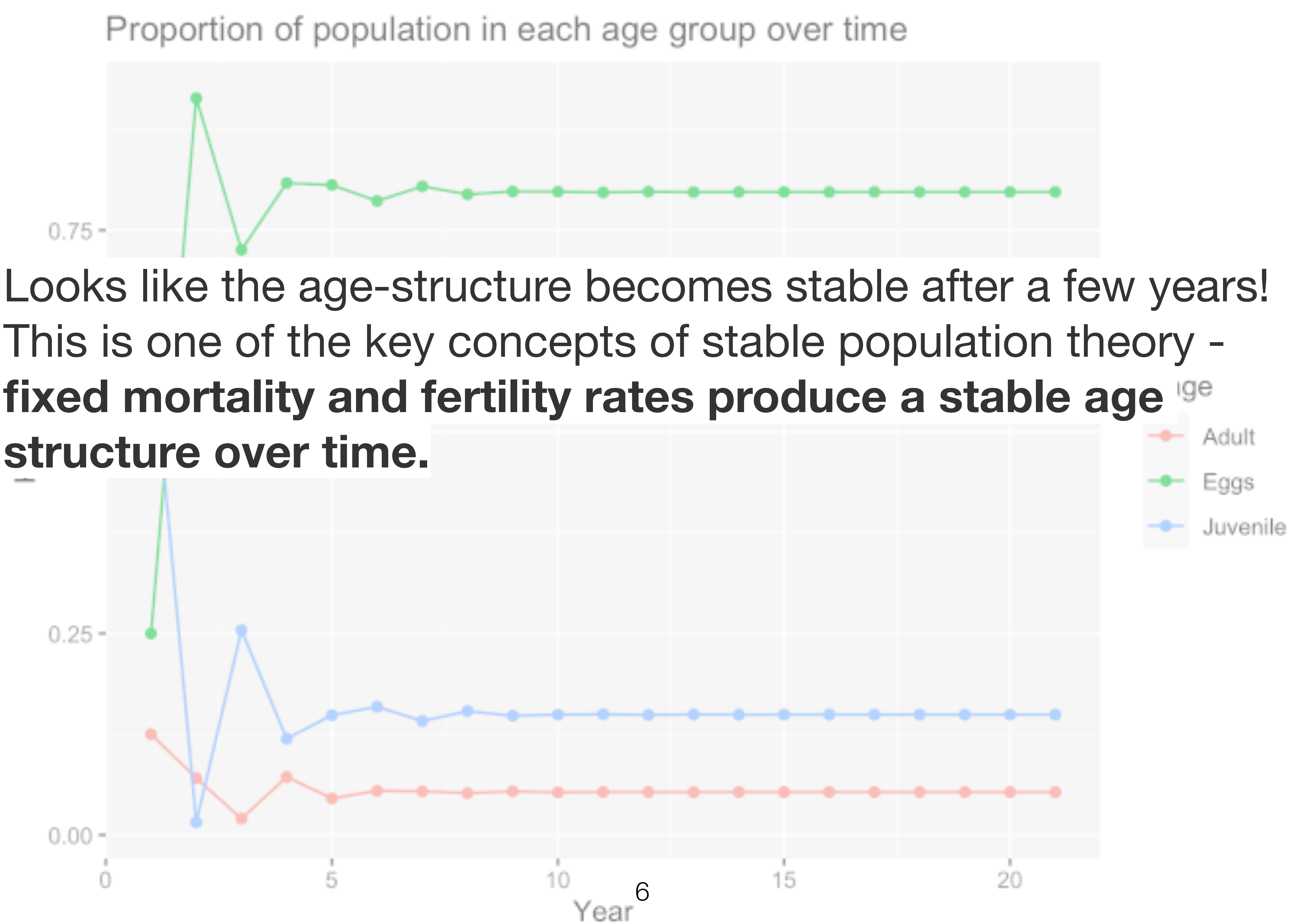




Quick recap of pre-workshop exercises

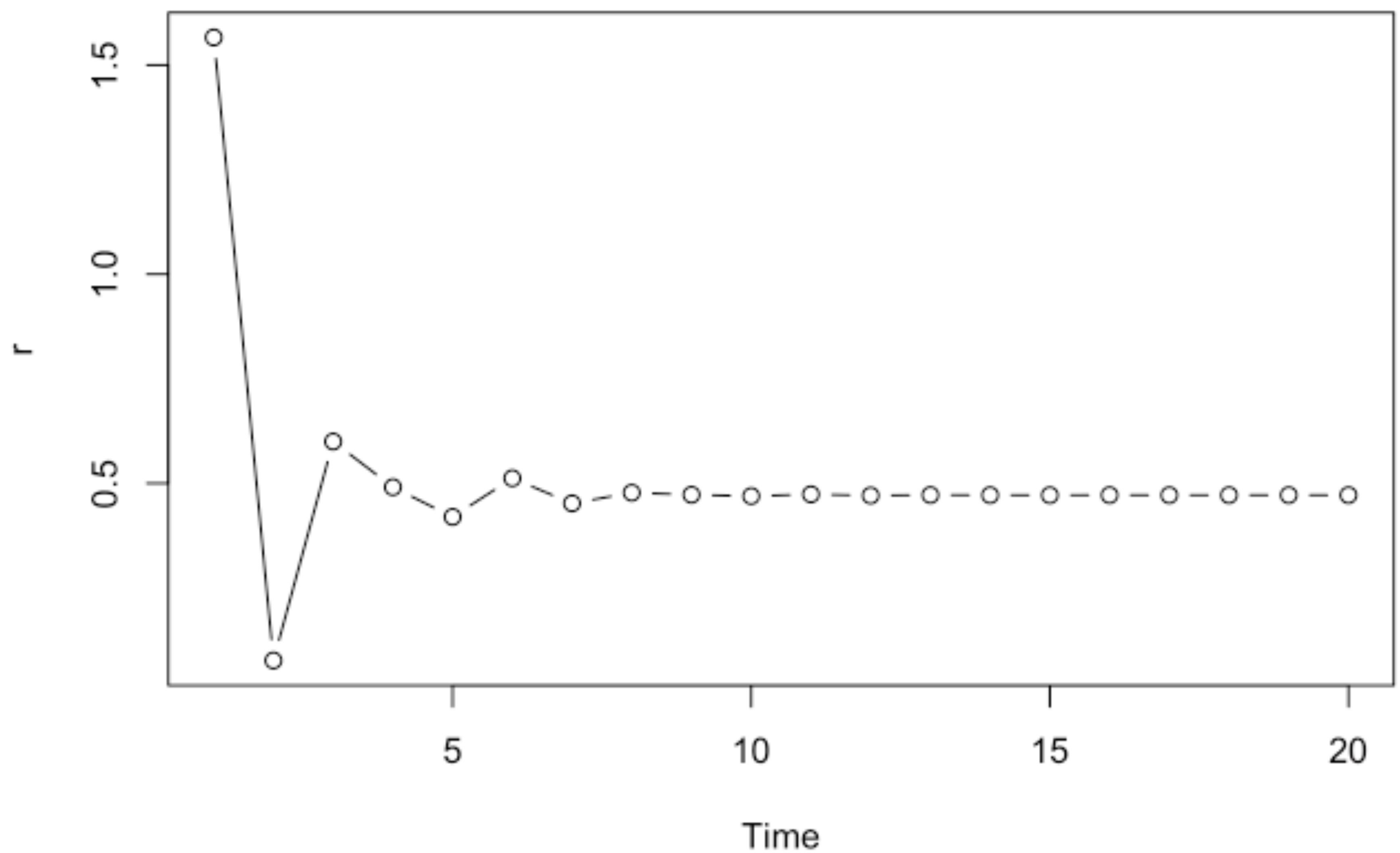


Quick recap of pre-workshop exercises



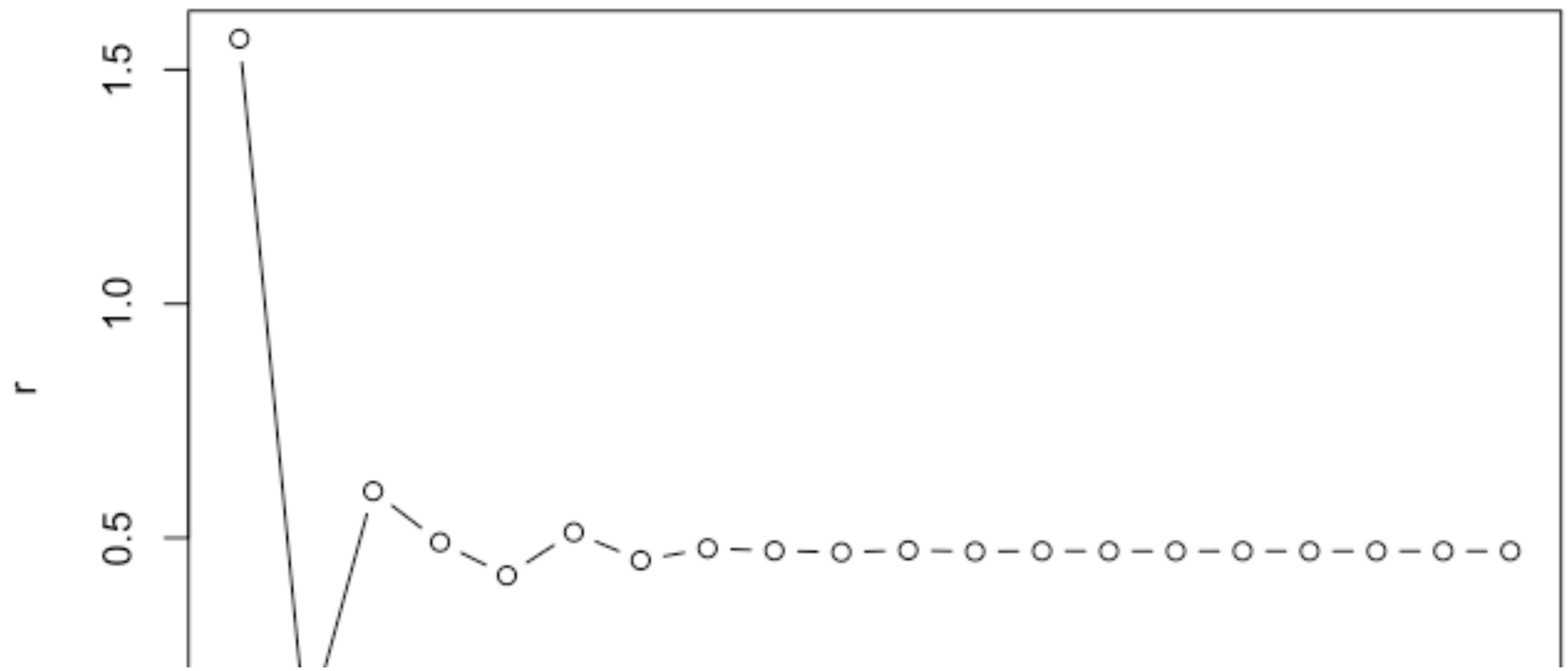
Looks like the age-structure becomes stable after a few years! This is one of the key concepts of stable population theory - **fixed mortality and fertility rates produce a stable age structure over time.**

Quick recap of pre-workshop exercises



Assuming exponential growth: K(t+1) = K (t)e^r where *r* is the annual growth rate

Quick recap of pre-workshop exercises



In the long run, the **population grows at an annual constant rate**! Moreover, we also saw that regardless of the characteristics of the starting population (i.e. age structure and size), if constant mortality and fertility rates are applied for a long time, **the population will eventually attain the characteristics (long-run *r* and stable age distribution) that are intrinsic to those constant rates**.

Stable population theory

The stable population model is used by demographers to demonstrate the long-term implications of maintaining short-term demographic patterns, and to identify the effects of the change in one parameter on the value of others.

An appreciation of the properties of stable populations helps us to understand the processes of destabilization following changes in fertility or mortality that are taking place all over the world in the 21st century.

Stable population theory: A simple example

Suppose births are growing exponentially: $B(t) = B(0) \times e^{rt}$

		Population alive					
Age (x)	probability of surviving from birth to age x (l_x)	year 0	year 1	year 2	year 3	year 4	year 5
0	1	1000					
1	0.75	0					
2	0.5	0					
3	0.25	0					
4	0	0					

Stable population theory: A simple example

Suppose births are growing exponentially: $B(t) = B(0) \times e^{rt}$

		Population alive					
Age (x)	probability of surviving from birth to age x (l_x)	year 0	year 1	year 2	year 3	year 4	year 5
0	1	1000	1000 e^r				
1	0.75	0	750				
2	0.5	0	0				
3	0.25	0	0				
4	0	0	0				

Stable population theory: A simple example

Suppose births are growing exponentially: $B(t) = B(0) \times e^{rt}$

		Population alive					
Age (x)	probability of surviving from birth to age x (l_x)	year 0	year 1	year 2	year 3	year 4	year 5
0	1	1000	1000 e^r	1000 e^{2r}			
1	0.75	0	750	750 e^r			
2	0.5	0	0	500			
3	0.25	0	0	0			
4	0	0	0	0			

Stable population theory: A simple example

Suppose births are growing exponentially: $B(t) = B(0) \times e^{rt}$

		Population alive					
Age (x)	probability of surviving from birth to age x (l_x)	year 0	year 1	year 2	year 3	year 4	year 5
0	1	1000	1000 e^{r}	1000 e^{2r}	1000 e^{3r}	1000 e^{4r}	1000 e^{5r}
1	0.75	0	750	750 e^{r}	750 e^{2r}	750 e^{3r}	750 e^{4r}
2	0.5	0	0	500	500 e^{r}	500 e^{2r}	500 e^{3r}
3	0.25	0	0	0	250	250 e^{r}	250 e^{2r}
4	0	0	0	0	0	0	0

Stable population theory: Population size over time

Let's look at change from K(t=year 3) to K(t=year 4):

$$\frac{K(t=4)}{K(t=3)} = \frac{1000e^{4r} + 750e^{3r} + 500e^{2r} + 250e^{r}}{1000e^{3r} + 750e^{2r} + 500e^{r} + 250} = e^{r}$$

		Population alive					
Age (x)	probability of surviving from birth to age x (l_x/l_0)	year 3	year 4	year 5	year 6	year 7	year 8
0	1	1000 e^{3r}	1000 e^{4r}	1000 e^{5r}	1000 e^{6r}	1000 e^{7r}	1000 e^{8r}
1	0.75	750 e^{2r}	750 e^{3r}	750 e^{4r}	750 e^{5r}	750 e^{6r}	750 e^{7r}
2	0.5	500 e^{r}	500 e^{2r}	500 e^{3r}	500 e^{4r}	500 e^{5r}	500 e^{6r}
3	0.25	250	250 e^{r}	250 e^{2r}	250 e^{3r}	250 e^{4r}	250 e^{5r}
4	0	0	0	0	0	0	0

Stable population theory: Population size over time

It looks like the age structure becomes stable from year 3 onwards. If we define year 3 as t = 0:

$$K(t) = K(0)e^{rt}$$

(you can check the math at home, for example by doing K(5)/K(3))

The population also grows exponentially!!

		Population alive					
Age (x)	probability of surviving from birth to age x (l_x/l_0)	year 3	year 4	year 5	year 6	year 7	year 8
0	1	1000 e^{3r}	1000 e^{4r}	1000 e^{5r}	1000 e^{6r}	1000 e^{7r}	1000 e^{8r}
1	0.75	750 e^{2r}	750 e^{3r}	750 e^{4r}	750 e^{5r}	750 e^{6r}	750 e^{7r}
2	0.5	500 e^{r}	500 e^{2r}	500 e^{3r}	500 e^{4r}	500 e^{5r}	500 e^{6r}
3	0.25	250	250 e^{r}	250 e^{2r}	250 e^{3r}	250 e^{4r}	250 e^{5r}
4	0	0	0	0	0	0	0

Stable population theory: Birth rate over time

Population also grows exponentially at the same constant rate as the number of births. **This means that the birth rate (births/total population) is constant over time.**

		Population alive					
Age (x)	probability of surviving from birth to age x (l_x/l_0)	year 3	year 4	year 5	year 6	year 7	year 8
0	1	1000 e^{3r}	1000 e^{4r}	1000 e^{5r}	1000 e^{6r}	1000 e^{7r}	1000 e^{8r}
1	0.75	750 e^{2r}	750 e^{3r}	750 e^{4r}	750 e^{5r}	750 e^{6r}	750 e^{7r}
2	0.5	500 e^{r}	500 e^{2r}	500 e^{3r}	500 e^{4r}	500 e^{5r}	500 e^{6r}
3	0.25	250	250 e^{r}	250 e^{2r}	250 e^{3r}	250 e^{4r}	250 e^{5r}
4	0	0	0	0	0	0	0

Stable population theory: Age structure over time

The number alive at each age changes from year to year - it grows exponentially at the rate, r. Because the total population is also growing exponentially at the same rate, the **proportion of the population in each age group becomes constant.**

		Population alive					
Age (x)	probability of surviving from birth to age x (l_x/l_0)	year 3	year 4	year 5	year 6	year 7	year 8
0	1	1000 e^{3r}	1000 e^{4r}	1000 e^{5r}	1000 e^{6r}	1000 e^{7r}	1000 e^{8r}
1	0.75	750 e^{2r}	750 e^{3r}	750 e^{4r}	750 e^{5r}	750 e^{6r}	750 e^{7r}
2	0.5	500 e^{r}	500 e^{2r}	500 e^{3r}	500 e^{4r}	500 e^{5r}	500 e^{6r}
3	0.25	250	250 e^{r}	250 e^{2r}	250 e^{3r}	250 e^{4r}	250 e^{5r}
4	0	0	0	0	0	0	0

Stable population theory: Age structure over time

To see this more clearly, we can express the ratio of population numbers in various age intervals to the number of births.

Using Year 4 as an example:

$$\frac{K(x=1, t=1)}{B(t=1)} = \frac{750e^{3r}}{1000e^{4r}} = e^{-r}\ \frac{l_1}{l_0} = e^{-r}\ l_1$$

Stable population theory: Age structure over time

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and

$$\frac{K(x=2, t=1)}{B(t=1)} = \frac{500e^{2r}}{1000e^{4r}} = e^{-2r}\ \frac{l_2}{l_0} = e^{-2r}\ l_2$$

Stable population theory: Age structure over time

To see this more clearly, we can express the ratio of population numbers in various age intervals to the number of births.

Using Year 4 as an example:

$$\frac{K(x=1, t=1)}{B(t=1)} = \frac{750e^{3r}}{1000e^{4r}} = e^{-r}\ \frac{l_1}{l_0} = e^{-r}\ l_1$$

and

$$\frac{K(x=2, t=1)}{B(t=1)} = \frac{500e^{2r}}{1000e^{4r}} = e^{-2r}\ \frac{l_2}{l_0} = e^{-2r}\ l_2$$

In general: $\frac{K(x,t)}{B(t)} = e^{-rx}\ l_x$ or $\boxed{K(x,t) = B(t)e^{-rx}\ l_x}$

Stable population theory: Age structure over time

Let's look at this equation more carefully:

$$\boxed{K(x,t)} = B(t)e^{-rx}\ l_x$$

Population aged x alive now at time t

Stable population theory: Age structure over time

Let's look at this equation more carefully:

$$K(x,t) = B(t)e^{-rx}\ l_x$$

Population aged x alive now at time t

Births that happened x years ago

$$B(t) = B(0)e^{rt}$$

$$B(t) = B(t-x)e^{rx}$$

$$B(2021) = B(2000)e^{21r};\ t = 2021, x = 21$$

$$B(t-x) = B(t)e^{-rx}$$

Stable population theory: Age structure over time

Let's look at this equation more carefully:

$$K(x,t) = B(t)e^{-rx} \, l_x$$

Population aged x alive now at time t	Births that happened x years ago	Probability of surviving to age x

Stable population theory: Age structure over time

Let's look at this equation more carefully:

$$K(x, t) = B(t)e^{-rx}\ l_x$$

Population aged x alive now at time t

Births that happened x years ago

Probability of surviving to age x

We will come back to this idea again later!

Stable population theory: Age structure over time

$$K(x,t) = B(t)e^{-rx}\ l_x$$

To get proportion at each age, divide by the total population, K:

$$\frac{K(x,t)}{K(t)} = c(x,t) = \frac{B(t)}{K(t)}e^{-rx}\ l_x$$

Thus: $c(x,t) = be^{-rx}\ l_x$

This is constant year to year since b and l_x are constant!

$$c(x) = be^{-rx}\ l_x$$

Stable population theory: Age structure over time

In a stable population, the total size of the population may change, growing or shrinking at a constant rate, but the number at every age changes at exactly the same rate, so that when it is expressed as a fraction of the total this proportion does not change over time.

The age composition, **c(x), is constant over time**, and can be expressed in terms of the birth rate, growth rate, and the life table survivorship function. It does not depend on the initial population size or age structure.

Recap

We have seen that a stable population emerges when:

1) births grow at a constant annual rate (we assumed exponential growth)

2) age-specific mortality rates (i.e. the life table) are constant

These conditions must prevail for a period - at least as long as the maximum age to which anyone survives.

Recap

The resulting stable population:

1) size changes at the same constant rate as the number of births

3) has a constant birth rate, death rate, and growth rate (b not necessarily equal to d; so r can be increasing, decreasing or constant)

2) has a constant proportion (but not necessarily number) of people alive at each age

Lotka's demonstration of the conditions necessary for a stable population

The mathematician Alfred Lotka showed that a stable population would emerge if:

1) age-specific fertility rates are constant (Note: the first condition in our simple example is replaced by this)

2) age-specific mortality rates (i.e. the life table) are constant

3) there is no net migration

Lotka's demonstration of the conditions necessary for a stable population

Lotka showed that a stable population would be produced by constant age-specific fertility and mortality rates applied over a long period of time.

He proved that constant age-specific fertility and mortality rates combine to produce the growth rate in the annual number of births, and of the entire population.

We won't go into the mathematical details of Lotka's proof but we will outline it.

Renewal equation

Lotka's proof is based on the **renewal equation (i.e. the idea that the population size today is dependent on births that happened in the past and the prevailing mortality rates).**

Consider a single-sex (female) projection:

$K(x, t)$ = # of women aged x at time t

$l(x, birth\ year)$ = survivorship to age x for cohort born in a certain year

$f(x, birth\ year)$ = fertility rate for x year olds who were born in a certain year (just considering female births; assume that we have multiplied fertility rates with f_{fab})

Renewal equation

$K(x, t) = B(t - x)\ l(x, t - x)$ = # of women aged x at time t

Let's break this down. We know that:

Number of x year olds today at time, t	**=**	**Number of 0 year olds x years ago at time, t-x**	**X**	**Probability of surviving to age x for the cohort born x years ago at time, t-x**

Renewal equation

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$$K(x, t) = K(0, t - x)\ l(x, t - x)$$

Renewal equation

$K(x, t) = B(t - x)\ l(x, t - x)$ = # of women aged x at time t

Let's break this down. We know that:

Number of x year olds today at time, t	=	Number of 0 year olds x years ago at time, t-x	X	Probability of surviving to age x for the cohort born x years ago at time, t-x

$$K(x, t) = K(0, t - x)\ l(x, t - x)$$

Births that happened x years ago at time, t-x

Renewal equation

$K(x, t) = B(t - x)\ l(x, t - x)$ = # of women aged x at time t

Let's break this down. We know that:

Number of x year olds today at time, t	**=**	**Number of 0 year olds x years ago at time, t-x**	**X**	**Probability of surviving to age x for the cohort born x years ago at time, t-x**

$$K(x, t) = K(0, t - x)\ l(x, t - x) = B(t - x)\ l(x, t - x)$$

Births that happened x years ago at time, t-x

Renewal equation

$K(x, t) = B(t - x)\ l(x, t - x)$ = # of women aged x at time t

Let's think about births (B) at time, t:

Babies born at time, t, to mothers who are x years old

$= K(x, t)\ f(x, t - x)$

Women aged x at time t

Fertility rates of x year olds who were born at time t-x

Renewal equation

$K(x,t) = B(t-x)\ l(x, t-x)$ = # of women aged x at time t

Let’s think about births (B) at time, t:

Babies born at time, t, to mothers who are x years old
$= K(x,t)\ f(x, t-x)$

Add up births to mothers of all ages to get total births at time, t:

$$B(t) = \int K(x,t)\ f(x, t-x)\ dx$$

Renewal equation

$K(x,t) = B(t-x)\ l(x,t-x)$ = # of women aged x at time t

Total births at time, B(t):

$$B(t) = \int K(x,t)\, f(x,t-x)\ dx$$

Substitute in above expression for $K(x,t)$:

$$B(t) = \int B(t-x)\ l(x,t-x)\, f(x,t-x)\ dx$$

In discrete time: $B(t) = \sum B(t-x)\ {}_nL_x(t-x)\ {}_nf_x(t-x)$

Renewal equation

$K(x,t) = B(t-x)\ l(x, t-x)$ = # of women aged x at time t

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$$B(t) = \int K(x,t)\, f(x, t-x)\ dx$$

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Renewal equation

$K(x, t) = B(t - x)\ l(x, t - x)$ = # of women aged x at time t

Total births at time, B(t):

$$B(t) = \int K(x, t)\ f(x, t - x)\ dx$$

Substitute in above expression for $K(x, t)$:

$$B(t) = \int B(t - x)\ l(x, t - x)\ f(x, t - x)\ dx$$

Births today

Women born in the past

Survived to age x

Had babies at this rate

This is the renewal equation! It relates births at time, t, to a stream of births in the past

Outline of Lotka's proof

Start with the renewal equation:

$$B(t) = \int B(t-x)\ l(x, t-x)\ f(x, t-x)\ dx$$

If mortality and fertility rates remain constant through time we don't need to keep track of the birth cohort (drop the t-x in the l and f function since those rates don't change over time):

$$B(t) = \int B(t-x)\ l(x)\ f(x)\ dx$$

Outline of Lotka’s proof

$$B(t) = \int B(t - x)\ l(x)\ f(x)\ dx$$

This is a homogenous integral equation. These equations can be solved by a process of trial and error, by looking for an expression for $B(t)$ which succeeds in equating the left-hand and right-hand sides of the equation.

Outline of Lotka's proof

$$B(t) = \int B(t-x)\ l(x)\ f(x)\ dx$$

Lotka (1939) showed that an exponentially growing birth series: $B(t) = B(0)e^{rt}$ is a solution to this renewal equation.

Outline of Lotka's proof

$$B(t) = \int B(t-x)\ l(x)\ f(x)\ dx$$

Lotka (1939) showed that an exponentially growing birth series: $B(t) = B(0)e^{rt}$ is a solution to this renewal equation.
$B(t) = B(0)e^{rt}$

or we can write $B(t) = B(t-x)e^{rx}$ where x is the number of years that have passed

Outline of Lotka's proof

$$B(t) = \int B(t-x)\ l(x)\ f(x)\ dx$$

Lotka (1939) showed that an exponentially growing birth series: $B(t) = B(0)e^{rt}$ is a solution to this renewal equation.
$B(t) = B(0)e^{rt}$

or we can write $B(t) = B(t-x)e^{rx}$ where x is the number of years that have passed

$B(t-x) = B(t)e^{-rx}$

Outline of Lotka’s proof

$$B(t) = \int B(t-x)\ l(x)\ f(x)\ dx$$

Substitute $B(t-x) = B(t)e^{-rx}$ into equation our renewal equation above.

Outline of Lotka's proof

$$B(t) = \int B(t-x)\ l(x)\ f(x)\ dx$$

Substitute $B(t-x) = B(t)e^{-rx}$ into equation our renewal equation above. If we plug this in, we get:

$$1 = \int e^{-rx}\ l(x)\ f(x)\ dx$$

This is the Lotka-Euler equation!

Intrinsic growth rate

$$1 = \int e^{-rx}\ l(x)\ f(x)\ dx$$

Although we won't prove it in this class, it can be shown that, given a set of $l(x)$ and $f(x)$, **there will always be a unique value of "r" such that the left-hand side is equal to 1**.

That value of r is the constant growth rate in the annual number of births, and is also the growth rate of the stable population.

It is referred to as the **"intrinsic growth rate"** of the stable population, because it is intrinsic to the mortality, $l(x)$, and fertility, $f(x)$, schedules that produced it. This means that the value of the growth rate, r, is not arbitrary - it is determined jointly by the age-specific fertility and age-specific mortality schedules.

Birth rate in stable population

The birth rate in a stable population is constant. This is because both the number of births and the total population size are growing at the same constant rate.

CBR = $b(t) = \dfrac{B(t)}{K(t)}$

At home, try deriving this equation from B(t) and K(t)!

Therefore, $b = \dfrac{1}{\int e^{-rx} \ l(x) \ dx}$

Since r and $l(x)$ are constant, the birth rate is also constant over time.

With discrete age groups: $b = \dfrac{1}{\sum e^{-rx} \ \frac{{}_nL_x}{l_0}}$

Stable age structure

Already saw in our simple example:

The age composition in a stable population, c(x), is constant over time, and can be expressed in terms of the birth rate, growth rate, and the life table survivorship function.

$$K(x,t) = B(t-x)\ l(x) = B(t)\ e^{-rx}\ l(x)$$

Proportion in the stable population (which is growing at rate, r) at age x:

$$c(x) = \frac{K(x,t)}{K(t)} = \frac{B(t)}{K(t)} e^{-rx}\ l(x) = b\ e^{-rx}\ l(x)$$

c(x) is constant over time in a stable population.

In discrete age groups (Wachter, page 229):

${}_nK_x(t) = B(t)\ e^{-rx}\ \dfrac{{}_nL_x}{l_0}$ and ${}_nc_x = b\ e^{-rx}\ \dfrac{{}_nL_x}{l_0}$

Characteristics of a stable population

These results show that the **age distribution, birth rate, death rate, and growth rate of a stable population are constant over time and are entirely determined by the age-specific fertility and age-specific mortality rates**.

Whatever the features of the population on which those fertility and mortality schedules are imposed, the population will eventually attain the characteristics “intrinsic” to the fertility and mortality schedules.

Implications of the stable population model

These results have powerful implications:

1) populations with unchanging vital rates, for a long-time, are stable (roughly true for a lot of human history, but not recent times).

2) Every populations' set of age-specific mortality and fertility rates imply an underlying stable equivalent population that would emerge if those rates remained constant for a long time. This gives us a sense of what current demographic parameters imply for long-run demographic prospects.

3) The relations established by Alfred Lotka provides a way for investigating how changes in one demographic parameter affects all others

Lab 1 Section I: Introduction to the data

Use fertility, mortality and initial population data from six countries.

Project population forward in time assuming unchanging mortality and fertility rates.

Investigate the implications of stable population theory.

In the afternoon, we will see how linear algebra and the eigendecomposition of the Leslie Matrix gives us equivalent results

Population projection using matrix algebra

The mechanics of the cohort component projection method can be compactly written in matrix notation. A lot of the work in this area can be attributed to Patrick H. Leslie (1945, 1948).

$K(1) = A \cdot K(0)$

$K(2) = A \cdot K(1) = A \cdot A \cdot K(0) = A^2 \cdot K(0)$

More generally, if A remains constant after t projection steps:

$K(t) = A^t \cdot K(0)$

Population projection using matrix algebra

$\text{K(t)} = \text{A}^{t} \cdot \text{K (0)}$

We saw in our pre-workshop exercises that when the Leslie matrix **A** is raised to a high enough power, the population age structure becomes constant and the population growth rate during each projection interval becomes constant.

This result is related to the stable population theory!

Population projection using matrix algebra

Moreover, matrix algebra (thanks to the Perron-Frobenius Theorem) offers an elegant way to derive the constant age distribution and the constant growth rate.

When a population has reached the steady state and has a constant age distribution, it must satisfy:

$K(t+1) = A \cdot K^s(t) = \lambda\ K^s(t)$ where $K^s(t)$ is now the unchanging stable age distribution

Population projection using matrix algebra

Moreover, matrix algebra (thanks to the Perron-Frobenius Theorem) offers an elegant way to derive the constant age distribution and the constant growth rate.

When a population has reached the steady state and has a constant age distribution, it must satisfy:

$K(t+1) = A \cdot K^s(t) = \lambda\ K^s(t)$ where $K^s(t)$ is now the unchanging stable age distribution

This might look familiar!
For two reasons!

Population projection using matrix algebra

$$K(t+1) = A \cdot K^s(t) = \lambda\ K^s(t)$$

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For two reasons!

First, for a stable population, we know that the population is growing exponentially.

So stable population theory tells, $\lambda = e^r$ (or $\lambda = e^{nr}$ where age groups are n years wide).

Population projection using matrix algebra

$$K(t+1) = A \cdot K^s(t) = \lambda\ K^s(t)$$

This might look familiar!
For two reasons!

Second, in linear algebra, eigendecomposition is a way to represent a matrix in terms of its eigenvectors and eigenvalues. For a diagonalizable square matrix, A, the vector v is an eigenvector of A if it satisfies:

$$\mathrm{A}\ \mathrm{v} = \lambda\ \mathrm{v}$$

where λ is the eigenvalue corresponding to the eigenvector v.

Eigendecomposition of the Leslie matrix

The Perron-Frobenius theorem guarantees that one eigenvalue will be positive and absolutely greater than all others. This is the dominant eigenvalue of the transition matrix.

Eigendecomposition of the Leslie matrix

It turns out that the dominant eigenvalue gives us the annual growth rate of the population: $\lambda = e^{nr}$ or $\dfrac{log(\lambda)}{n} = r$

Eigendecomposition of the Leslie matrix

It turns out that the dominant eigenvalue gives us the annual growth rate of the population: $\lambda = e^{nr}$ or $\dfrac{log(\lambda)}{n} = r$

The right eigenvector associated with the dominant eigenvalue gives us the stable age distribution: v = K^s

The left eigenvector associated with the dominant eigenvalue gives us the reproductive value, which tells us something about the relative contributions of each age class to the next generation.

Lab 1 Section II: Population Projections

Use fertility, mortality and initial population data from six countries.

Project population forward in time assuming unchanging mortality and fertility rates.

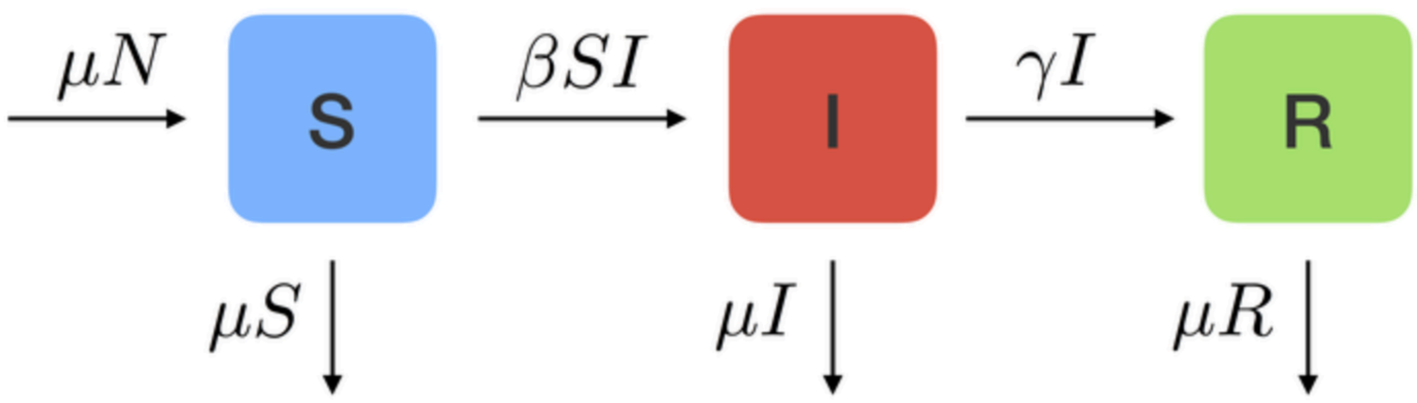
Investigate the implications of stable population theory.

In the afternoon, we will see how linear algebra and the eigendecomposition of the Leslie Matrix gives us equivalent results

Infectious disease models

There are parallels between models of population projection and models of infectious disease transmission.

The Susceptible-Infected-Recovered (SIR) model consists of three states (rather than ages) and captures the transition between these states.



Infectious disease models

The **next generation matrix** is a square transition matrix (similar to the Leslie matrix) that describes the transition from one state to another.

The **dominant eigenvalue of the next generation matrix is the basic reproduction number,** R_0, which is the expected number of secondary infections from one infected person in a fully susceptible population.

Infectious disease models

If we have an **age and disease stage structured population**, for a respiratory pathogen such as SARS-CoV-2 we can define the next-generation matrix as:

$\mathbf{NGM = D_u \, . \, C \, . \, D_{dI}}$

$\mathbf{D_u}$ = diagonal matrix with diagonal entries u_i representing the probability of a successful transmission for age group i, given contact with an infectious individual

C = contact matrix, where the entries c_{ij} represents the average number of age-j individual than an age-i individual contacts per day

$\mathbf{D_{dI}}$ = diagonal matrix with diagonal entries dI equal to the infectious period

Infectious disease models

If we have an **age and disease stage structured population**, for a respiratory pathogen such as SARS-CoV-2 we can define the next-generation matrix as:

R_0 is the dominant eigenvalue of NGM

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Lab 1 Section III: Social Contact Matrices and Infectious Disease Models

In the final part of the lab exercise, we will look at data from the Berkeley Interpersonal Contact Study (BICS) which has been collecting data on interpersonal contact over the course of the pandemic.

This allows us to generate age-structured contact matrices for the US.

We will use these contact matrices, and estimates of D_u and D_{dI} from the literature, to estimate R_0 for COVID-19.