Entropy of the life table: the effects of Covid on life expectancy

BFDW, Day 2, Lecture 2

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May 2021

- How many people have studied life table?
- Discrete or continuous?
- Continuous notation makes math easier

Life table definitions, in continuous time

Hazard, instantaneous death rate

$$h(a) = -\frac{\ell'(a)}{\ell(a)} = -\frac{d}{da}\log \ell(a)$$

Survival, fraction still alive

$$\ell(a) = e^{-\int_0^a h(x)\,dx}$$

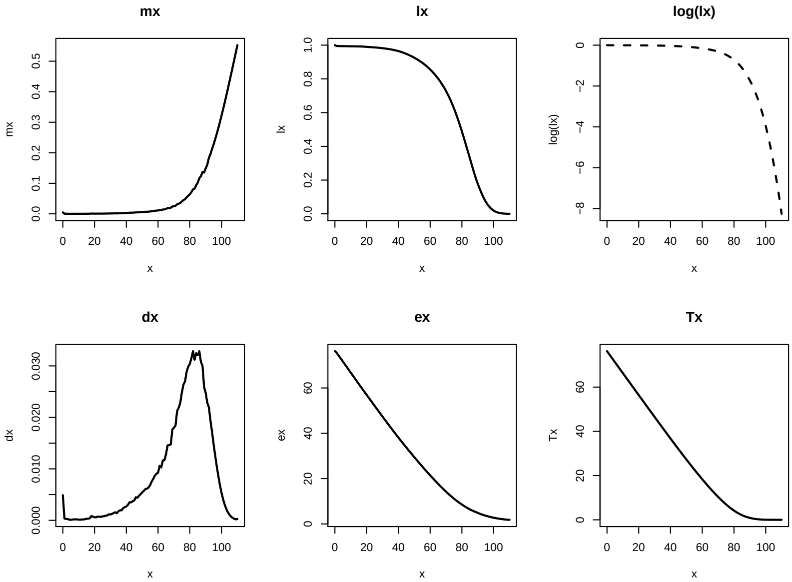
Life expectancy at age a

$$e(a) = \int_a^\infty \ell(x)\,dx \,/\, \ell(a)$$

and at birth, age 0,

$$e(0) = \int_0^\infty \ell(a)\,da$$

Life table functions, Taiwan Males 2010 (from HMD)



Our Question: How much does Covid reduce life expectancy?

Answer is related to a classic question in formal demography: how does same *proportional* change in hazards at all ages affect life expectancy at birth?

- Because Covid's effect is roughly proportional
- We can use Keyfitz entropy $\mathcal{H}$

Proportionality

$$h(a) = (1 + \delta) h_{base}(a)$$

We want to know how changing hazards influences life expectancy

Write life expectancy as function of δ,

$$e_0(\delta) = \int \ell(a)^{1+\delta} da$$

How much does life expectancy change when hazards change proportionally?
Around $\delta = 0$,

$$\frac{d}{d\delta} e_0(\delta) = \int [\log \ell(a)] \ell(a) \, da$$

(If time, let's let Tom derive this.)

Positive or negative?

$$\frac{d}{d\delta}e_0(\delta) = \int [\log \ell(a)]\ell(a)\, da$$

Keyfitz entropy $\mathcal{H}$

Keyfitz $\mathcal{H}$ tells *proportional decrease* in life expectancy per *proportional increase* in hazards. So,

- He puts a minus sign in front of derivative
- He makes relative to starting life expectancy

Thus,

$$\mathcal{H} \equiv \left. \frac{-\frac{d}{d\delta} e_0(\delta)}{e_0} \right|_{\delta=0} = \frac{-\int_0^{\omega} \ell(x) \log \ell(x) \, dx}{e_0}$$

Properties of entropy

Keyfitz's entropy measure $\mathcal{H}$, makes change in e_0 proportional

$$\mathcal{H} = \frac{-\int_0^\omega \ell(x) \log \ell(x)\, dx}{e_0}$$

Properties

- Positive or negative?
- What if everyone dies at age 100?
- What if hazards are constant?
- Why "entropy"?
- A more intuitive interpretation?

(Optional:) entropy as effect of letting people die twice

$$\text{numerator of } \mathcal{H} = -\int_0^{\omega} \ell(x) \log \ell(x)\, dx$$

- Rewrite $\log \ell(x)$ in terms of *hazards*

(Optional:) entropy as effect of letting people die twice

$$\text{numerator of } \mathcal{H} = -\int_0^{\omega} \ell(x) \log \ell(x)\, dx$$

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- Reverse order of integration

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- Stare at and see life table functions

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$$\mathcal{H} = -\frac{\int_0^{\omega} e(x)\ell(x)h(x) \, dx}{e(0)}$$

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(Why is this like letting people die twice?)

Application to Covid

For applications, one can rewrite as our equation (2) on "cheatsheet"

$$\frac{\Delta e_0}{e_0} \approx -\mathcal{H}\delta$$

Or,

$$\Delta e_0 \approx -\mathcal{H}\delta e_0$$

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For example, assume

- $\delta = 1/3$, (≈ 1 million deaths in US)
- $e_0 = 80$
- $\mathcal{H} = 0.15$

$$\Delta e_0 = -\delta \mathcal{H} e_0 =$$

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$$\Delta e_0 = -\delta\mathcal{H}e_0 = (1/3)(0.15)(80) = 4 \text{ years}$$

What mistake did I make?

Breakout Room Exercises

A Use full life table calculation to calculate effect of 1/6 increase in US mortality (like in pre-workshop exercises)

B Use entropy approximation, and compare.

C Use more realistic age-distribution for Covid

Discussion of Exercises

- Does doubling mortality half life expectancy?
- Is entropy approximation pretty accurate? Why might it be inexact?
- Is effect of Covid on life expectancy larger or smaller than proportional change in hazards (with same number of deaths)? Why?

What other questions do you have?

After lunch

- Life expectancy critics
- An alternative measure: person years lost
- Elegance $\neq$ relevance?